THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-seventh meeting of the Society was held in New York City on Saturday, February 29, 1908, extending through the usual morning and afternoon sessions. The attendance included the following thirty-four members:

Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Professor F. N. Cole, Professor L. P. Eisenhart, Mr. G. W. Hartwell, Dr. A. M. Hiltebeitel, Professor W. H. Jackson, Professor Edward Kasner, Mr. E. H. Koch, Mr. W. C. Krathwohl, Dr. G. H. Ling, Mr. L. L. Locke, Professor Max Mason, Mr. A. R. Maxson, Professor H. B. Mitchell, Dr. R. L. Moore, Mr. H. W. Reddick, Professor L. W. Reid, Mr. L. P. Siceloff, Mr. F. H. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. W. M. Strong, Dr. Elijah Swift, Professor H. D. Thompson, Miss M. E. Trueblood, Professor J. M. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor A. G. Webster, Professor H. S. White, Professor J. W. Young.

The President of the Society, Professor H. S. White, occupied the chair, being relieved at the afternoon session by Professor P. F. Smith. The Council announced the election of the following persons to membership in the Society: Mr. E. G. Bill, Yale University; Mr. C. H. Currier, Brown University; Mr. W. S. Pemberton, State Normal School, Edmond, Okla.; Professor S. W. Reaves, University of Oklahoma; Mr. L. L. Silverman, University of Missouri; Mr. W. M. Smith, Lafayette College; Mr. H. W. Stager, University of California. Twelve applications for membership in the Society were received.

The By-Laws of the Society were amended to provide that the appointment of members of the Editorial Committee of the *Transactions* should be made at the April meeting of the Council and that each new member should take office on the following October 1.

The usual informal dinner was arranged for the evening, and was attended by twelve members.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On the numerical factors of certain arithmetic forms."

(2) Professor R. D. CARMICHAEL: "On the remainder term in a certain development of $f(a+x)$."

(3) Dr. F. R. SHARPE: "The Lorentzian transformation and the radiation from a moving electron."

(4) Professor VIRGIL SNYDER: Normal curves of genus 6 and their groups of birational transformations."

(5) Professor J. W. YOUNG: "The geometry of chains on a complex line."

(6) Professor J. W. YOUNG: "A fundamental invariant of discontinuous ζ-groups defined by a normal curve of order n in space of n dimensions."

(7) Professor E. B. VAN VLECK: "On non-measurable point sets."

(8) Professor L. E. DICKSON: "On higher congruences and modular invariants."

(9) Professor MAX MASON: "Note on Jacobi's equation in the calculus of variations."

(10) Dr. G. W. HILL: "Subjective geometry."

(11) Professor G. A. BLISS: "A method of deriving Euler's equation by means of an invariant integral."

(12) Professor C. N. HASKINS: "On the second law of the mean."

(13) Professor EDWARD KASNER: "The contact transformations of mechanics."

(14) Professor EDWARD KASNER: "The plane sections of an arbitrary surface."

(15) Professor G. A. MILLER: Note on the periodic decimal fractions."

(16) Dr. F. R. SHARPE: "The inner force of a moving electron."

(17) Dr. W. H. ROEVER: "Brilliant points of curves and surfaces."

(18) Mr. FRANK IRWIN: "Transformations of the elements $x, y, y', \cdots, y^{(k)}$ that carry a union of such elements over into a union."

Mr. Irwin's paper was communicated to the Society through Professor Bouton. In the absence of the authors, Professor Haskins's paper was read by Professor Mason, Dr. Sharpe's second paper was read by Professor Snyder, and Dr. Sharpe's first paper and the papers of Professor Carmichael, Professor

Van Vleck, Professor Dickson, Dr. Hill, Professor Miller, Dr. Roever, and Mr. Irwin were read by title.

Professor Van Vleck's paper will appear in the April *Transactions*. The papers of Professor Dickson, Professor Mason, and Dr. Hill appeared in the April BULLETIN. Professor Young's second paper and Dr. Sharpe's second paper appear in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael obtains several propositions concerning the numerical factors of arithmetic forms determined thus: Let $Q_n(x) = 0$ be the equation whose roots are the primitive nth roots of unity. In $Q_n(x)$ put $x = \alpha/\beta$, α and β being relatively prime positive integers and $\alpha > \beta$. Multiply this expression by β raised to a power equal to the degree of $Q_n(x)$. A study is made of the factors of the resulting forms. Several propositions are proved which are generalizations of those obtained by Dickson in an article "On the cyclotomic function" (*American Mathematical Monthly*, volume 12, pages 86–89).

2. The series which Professor Carmichael studies in this paper, with the remainder term which he finds, is as follows:

$$f(a+x) = f(a) + \frac{x}{m}\left[f'\left(a+\frac{x}{m}\right) + f'\left(a+\frac{2x}{m}\right) + \cdots \right.$$

$$\left. + f'\left(a+\frac{m-1}{m}x\right) + f'(a+x)\right] - \frac{x}{m\cdot 2}[f'(a+x) - f'(a)]$$

$$- \frac{B_1 x^2}{m^2\cdot 2!}[f''(a+x) - f''(a)] + \frac{B_2 x^4}{m^4\cdot 4!}[f^{\mathrm{IV}}(a+x) - f^{\mathrm{IV}}(a)]$$

$$- \cdots + (-1)^n \frac{B_n x^{2n}}{m^{2n}\cdot(2n)!}[f^{(2n)}(a+x) - f^{(2n)}(a)].$$

$$+ x^{2n+2}\left[\frac{1}{(2n+2)!} + \frac{1}{2m(2n+1)!} + \frac{B_1}{m^2\cdot 2!(2n)!}\right.$$

$$\left. + \frac{B_2}{m^4\cdot 4!(2n-2)!} + \cdots + \frac{B_{n-1}}{m^{2n-2}\cdot(2n-2)!\,4!} + \frac{B_n}{m^{2n}\cdot(2n)!\,2!}\right]$$

$$\times [f^{(2n+2)}(a+\phi_1 x) - f^{(2n+2)}(a+\phi_2 x)],$$

B_1, B_2, $\cdots$ being Bernoulli's numbers, and ϕ_1 and $\phi_2 < 1$ and > 0.

This series, without remainder term, was given by S. A. Corey in the *Annals of Mathematics*, second series, volume 5, number 4 (July, 1904).

3. The field due to a moving charge has been discussed by many writers: Langevin in particular has given an elegant investigation of the field due to a moving electron. Poincaré has partially solved Langevin's problem by the use of the Lorentzian transformation. In Dr. Sharpe's paper the Lorentzian transformation is applied to the definite integral expressions for the scalar and vector potentials of the field instead of to the differential equations of the field. By aid of the transformation the problem of Langevin is reduced to the simpler problem where the velocity of the electron vanishes at the instant considered. By transforming back again, the solution of Langevin's problem is obtained with a new form for the magnetic force.

4. In Professor Snyder's paper, curves of genus 6 are defined as the partial intersection of five particular quadric spreads in space of five dimensions, R_5, and one arbitrary spread. By proper projection, this curve of order 10 can be reduced to a plane sextic with four double points. The coordinate spaces become adjoint cubics, in terms of which an equation of any sextic of this type can be written. Every possible birational transformation of this curve into itself can be generated by four linear operators of order two. The most interesting groups are G_{120}, to which a sextic with four double points at the vertices of a quadrangle belongs, a G_{150} leaving invariant a non-singular quintic (not reducible to a sextic), and a cyclic G_{26} of a certain hyperelliptic curve. The study of the non-singular quintic from this standpoint involves that of the Veronese surface in R_5.*

5. A chain in the ordinary complex projective geometry may be defined analytically as any class of points projective with the real points of a line, from which follows that there is one and only one chain through three given points of a line. With the usual representation of complex numbers on a sphere a

* Wiman, *Bihang till k. Sven. Vet. Akad. Handlingar*, vol. 21 (1895), two articles. Clebsch, Geometrie, vol. 1, p. 695 ff. Segre, *Ann. di mat.* (2), vol. 22 (1894).

chain is simply a circle. A chain may also be defined directly in terms of three points on a line by means of the notion of a net of rationality on a line and order relations, as was done in a paper presented to the Society by Professors Veblen and Young at its last meeting. In his first paper Professor Young develops some of the fundamental properties of the chains of a line by purely synthetic methods, the results being well known in the theory of functions of a complex variable when the chains are identified with the circles on the sphere of the complex variable, or the Argand plane. The proofs are in many cases much simpler than the usual analytic proofs. Moreover it appears important to emphasize the fact that the function-theoretic theorems in question form an elementary chapter in projective geometry.

The system of chains through two given points M, N is called the system $T(M, N)$. Through any point P distinct from M, N there is one and only one chain cutting each of the chains through M, N in points harmonic with M, N. Every such chain is said to be "about" M, N; the system of all chains about M, N is denoted by $A(M, N)$. The two systems T and A are fundamental in the classification of projectivities with distinct double points. If a chain is about two points M, N, these points are said to be conjugate with respect to the chain, and one is the inverse of the other. Then if every point on a chain is defined as its own inverse, the existence and uniqueness of the inverse of any point of a line with reference to a given chain follows. Two chains are defined as orthogonal if one contains two points conjugate with respect to the other. The projectivities on a line are then classified, with reference to their invariant figures, into involutoric, elliptic, hyperbolic, loxodromic, and parabolic projectivities.

11. In the paper of Professor Bliss a method of deriving Euler's equation in the calculus of variations by means of the theory of invariant integrals of the form

$$\int \left\{ A(x, y) + B(x, y) \frac{dy}{dx} \right\} dx$$

is exhibited. The method permits one to derive very simply the result of Du Bois-Reymond and Hilbert, that for a regular problem the minimizing curve can not have corner points where

the slope is discontinuous. A considerable simplification in the presentation of the whole theory of the simplest problem in the plane is also effected, because the notion of an invariant integral is one which can be frequently applied in the derivation of the further conditions.

12. The second law of the mean is fundamental in the classical Dirichlet discussion of Fourier's series. The fundamental theorem of Fourier's constants, however, as proved by de la Valleé Poussin and Hurwitz, does not involve the second law of the mean. In Professor Haskins's note it is shown that the second law of the mean is a simple consequence of the fundamental theorem of Fourier's constants.

13. In the *Leipziger Berichte* for 1889 Lie studied those infinitesimal contact transformations whose characteristic function is of the form $\Omega(x, y)\sqrt{1 + p^2}$ and indicated their importance in dynamics. In the present note Professor Kasner shows that the alternant of any two transformations of this type is necessarily a point transformation. The result holds for any number of variables and may be applied to many allied theories.

14. In connection with any surface S we may consider the curves cut out by the ∞^3 planes of space. By projecting these orthogonally or centrally upon a fixed plane we obtain the triply infinite systems which are investigated in Professor Kasner's paper. Some of the properties derived are analogous to those which hold for dynamical trajectories: in particular the focal locus is circular for both types. The general classes are however distinct, and overlap only when the surface S is cylindrical or conical. The developable surfaces give rise to especially simple results. The theory worked out belongs to general projective geometry.

15. The group G formed by the $\phi(n)$ numbers which are less than n and prime to n, when they are combined with respect to multiplication modulo n, has been called by H. Weber the most important example of an abelian group of finite order. The object of Professor Miller's note is to point out how several fundamental theorems relating to the periods of decimal fractions may readily be proved by means of this group. Assuming $n > 1$ and prime to 10, it results that 10, or

its residue modulo n, is a number in G and that the powers of 10 generate a cyclic subgroup K whose order k is the length of the period of $1/n$, since K includes 1 and multiplying a decimal fraction by 10 simply moves its decimal point one place to the right. The operators of K have therefore the characteristic property that they simply affect the position of the decimal point of any number divided by n with which they are combined under G, moving it respectively $1, 2, \cdots, k$ places to the right. If these operators are applied a second time in order they will move this point respectively $k+1, k+2, \cdots, 2k$ places to the right. As the k distinct numbers modulo n obtained in the second case are the same as those obtained in the first, and each of the $\phi(n)$ numbers divided by n gives rise to a different decimal fraction it results that k is the length of the period of *every* number of G divided by n.

From the fact that G always involves -1 it follows that the numbers of G may be divided into pairs differing only with respect to sign, and hence the mantissa of the one divided by n may be obtained by subtracting from g each digit in the mantissa of the other divided by n. If K involves -1, the mantissas of the two numbers differing only in sign divided by n may be obtained from each other by a cyclic permutation and hence the entire period of such a number may be obtained by subtracting from g each digit of half the period. When the index of K under G is odd, in particular when 10 is a primitive root of n, K must involve -1 and hence half the period of every number of G divided by n may be obtained by subtracting from g the digits of the other half in order. The index of K under G is the number of distinct periods in the fractions obtained by dividing by n all the numbers which are prime to n. If the mantissa of a number of G divided by n were composed of g's, that of the other number of the pair would be composed of o's; *i. e.*, the otherwise evident fact that g cannot constitute the period of a proper fraction in decimal form may be regarded as a special case of the given theorems. With the exception of the periods o and g, any arbitrary set of numbers may evidently be the period of a proper fraction.

17. In this paper, which is an extension of one reported in the BULLETIN (series 2, volume 8 (1901–02), page 279) and published in the *Annals of Mathematics* (series 2, volume 3 (1901–02), page 113), Dr. Roever begins with a description of

the optical phenomenon which suggested the problems considered. A brilliant point of a curve c with respect to two congruences is defined as a point P of c at which the curves of the two congruences which pass through P make equal angles with c at P. A system of three congruences is then considered, and the condition found that a point P should be a brilliant point of the curve of one of these which passes through P, with respect to the other two. The condition consists in the vanishing of a certain expression Ω, which involves the functions that appear in the systems of differential equations of the congruences. It is found that the expression Ω possesses a certain kind of invariance. Thus, if each of the three congruences represented in Ω be replaced by a certain congruence involving an arbitrary function, the expression Ω formed from the new congruences differs from that formed from the old ones by a non-vanishing factor. Therefore the equation $\Omega = 0$ is the locus of the brilliant points of each of an infinite number of congruences with respect to each of an infinite number of pairs of congruences. Two special kinds of brilliant point are considered in great detail. In terms of either of these, the brilliant point of a surface with respect to a pair of congruences, may be defined by considering a congruence normal to the surface. It is shown that under certain restrictions brilliant points of both curves and surfaces, may be regarded as points of contact. It is also shown that optical interpretations of one or another kind may be given to most of the results. Numerous examples illustrate the results.

18. Mr. Irwin shows that the only transformations of the elements $(x, y, y', \cdots, y^{(k)})$ carrying a union of these elements over into a union are extended contact transformations.

F. N. COLE,
Secretary.

THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE thirteenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, February 29, 1908, at Stanford University. Professor Hoskins presided. The following thirteen members were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Mr. Joseph Lipke, Dr. J. H. McDonald, Professor W. A. Manning, Professor Irving Stringham, Mr. W. H. Stager, Mr. J. D. Suter, Professor S. D. Townley.

The following papers were read at this meeting:

(1) Professor D. N. LEHMER: "A discussion by synthetic methods of the covariant conic of two given conics."

(2) Mr. JOSEPH LIPKE: "Ein Fehler im König'schen Beweise des Reciprocitätsgesetzes in der Theorie der quadratischen Reste."

(3) Professor G. A. MILLER: "Generalization of the positive and negative numbers." (By title.)

(4) Professor L. M. HOSKINS: "A general diagrammatic method of representing propositions and inference in the logic of classes."

(5) Dr. J. H. MCDONALD: "On certain types of continued fractions considered from a common point of view."

(6) Mr. J. D. SUTER: "On the surface $F_1(u) = u/(1 - u^4)$, $F_2(v) = v/(1 - v^4)$" (preliminary report).

(7) Professor H. F. BLICHFELDT: "On a certain basis of geometry" (preliminary report).

(8) Professor L. M. HOSKINS: "General algebraic solutions in the logic of classes."

Abstracts of the papers follow below in the order as given above.

1. Professor Lehmer gave a discussion by synthetic methods of the problem: To find the locus of a point in the plane such that the four tangents from it to two fixed conics shall be harmonic. The proof is made to depend on the theorem: Given a point row of the second order and a pencil of rays of the second order projective to it, at most four rays of the pencil

will pass through the points on the point row which correspond to them.

2. In the *Acta Mathematica*, volume 22, pages 181–191, Professor König gives a proof of the theorem of reciprocity in the theory of quadratic residues, a proof by complete induction but without the use of the famous gaussian lemma, according to which, if p is a prime of the form $8n + 1$, there always exists a prime q less than $2\sqrt{p} + 1$ such that p is a quadratic non-residue of q. In this note, Mr. Lipke points out an error in Professor König's work which invalidates the latter's proof of the theorem, and also illustrates by an example that there is something radically wrong with the method of proof adopted.

3. Calling the totality of the ordinary complex numbers which have α_0 for their amplitude the α_0-numbers, or the α_0 ray of numbers, Professor Miller considers the various sets of rays in regard to group properties with respect to the fundamental operations of arithmetic. In accord with this general nomenclature, the positive numbers are called 0-numbers and the negative ones are called π-numbers, the number 0 being a special 0-number. Each of two rays whose amplitudes differ by π is said to be the extension of the other, and it is observed that a ray and its extension always form a group with respect to addition or subtraction, but that no other finite number of rays has this property. The common rule for adding a positive and a negative number is included in the following: To find the sum of a number on a ray and a number on its extension, take the difference of their absolute values and prefix the angle of the one which has the larger absolute value. If a set of numbers forms a group with respect to multiplication or division, it involves either only one number from each ray represented in the set or it involves an infinite number of numbers from each one of these rays; and if a finite number of rays forms a group with respect to these operations, it includes the ray of 0-numbers. The 0-numbers and the π-numbers constitute the only finite number of rays forming a group with respect to both of the operations addition and multiplication, and this is one reason why these numbers are of such special importance. If a finite number of rays forms a group with respect to multiplication or division, these rays as units form a cyclic group. The paper will appear in the *American Mathematical Monthly*.

4. The character of the general solutions referred to by Professor Hoskins is illustrated by what may be called the generalized syllogism, which may be stated as follows: Given any two propositions involving x, y and y, z respectively, it is required to infer a proposition involving x, z. The three propositions being written in the most general equational form, according to Boole's symbolism, the result of the solution is so to express the coefficients that when the premises are given it is seen at a glance which (if any) of the coefficients in the conclusion are determined. The method of solution consists in regarding each of the three propositions forming the premises and the conclusion as derived by elimination from the general proposition involving x, y, z. The general result includes both universal and particular (or existential) propositions.

5. This paper defines by means of certain inequalities the manner in which the development in a continued fraction may be formed. These inequalities lead to three distinct types. Dr. McDonald has investigated the rapidity of convergence of each type.

6. Mr. Suter discussed certain properties of the surface $F_1(u) = u/(1-u^4)$, $F_2(v) = v/(1-v^4)$ by means of a system of equations of the form

$$(a)\ x = \phi_1(\xi, \eta), \quad (b)\ y = \phi_2(\xi, \eta), \quad (c)\ z = \phi_3(\xi, \eta),$$

found by substituting the given functions in Weierstrass's formulas for expressing the rectangular coordinates of any point on a minimal surface in terms of the imaginary parameters $u = \xi + i\eta$, $v = \xi - i\eta$. He showed that every pair of values (ξ, η) which satisfies the hyperbola $\eta^2 - 2\xi\eta\alpha^{-1} - \xi^2 + 1 = 0$, gives by substitution in (a) and (c), points on the section of the surface with the plane $y = -\frac{1}{2}\tan^{-1}\alpha$; and suggested in this manner a method for modelling the surface; namely, by constructing sections in planes parallel to the xz plane.

7. Only a finite number of types of trigonometries are possible under the following set of assumptions made in regard to a system of an infinite number of points (which we call a plane):

(1) With every pair of points a, b of the system belong three numbers; one we call the distance, (ab) or (ba); and the other two directions, $\overline{ab}$ and $\overline{ba}$ respectively.

(2) The existence of a trigonometry is assumed. If a, b, c are any three points of the system, we postulate a relation of the form

$$\overline{ba} - \overline{bc} = f[(ab), (bc), (ac)],$$

where $f(\alpha, \beta, \gamma)$ satisfies certain conditions as to continuity, etc., when α, β, γ are considered as independent variables.

(3) If c is a constant, and α_1, α_2, β_1, β_2 independent variables, the equations

$$f(\alpha_1, \beta_1, \gamma) + f(\beta_1, c, \beta_2) + f(c, \alpha_1, \alpha_2) = 0,$$
$$f(\alpha_2, \beta_2, \gamma) + f(\beta_2, c, \beta_1) + f(c, \alpha_2, \alpha_1) = 0,$$

can be solved for γ and the solutions are identical.

Professor Blichfeldt remarked that the conditions (2) and (3) have evident geometric meanings in the ordinary euclidean and non-euclidean planes.

8. The representation proposed by Professor Hoskins depends upon an analogy which may be explained by reference to the case of three class terms x, y, z. Let the three pairs of opposite faces of a cube be regarded as representing (x, x'), (y, y'), (z, z') respectively (x' being written for not-x, etc.); then the domain common to any two of these six classes may be represented by the edge determined by the intersection of their representative planes, while the point of intersection of any three of the six planes may represent one of the eight subclasses xyz, xyz', $\cdots$. If each face of the cube is marked with the corresponding class symbol, the relations among the subclasses are seen at a glance. A convenient plane diagram is obtained by drawing a projection of a cube, with a circular space at each vertex which in any particular case can be marked in accordance with the import of a given proposition. A symmetric representation for the case of n primary class terms would require space of n dimensions. But diagrams for practical use may be obtained by repeating the diagram for three terms. Thus the case of six terms would be represented by eight similar cubes, each at a vertex of a larger cube.

W. A. MANNING,
Secretary of the Section.

A FUNDAMENTAL INVARIANT OF THE DISCONTINUOUS ζ-GROUPS DEFINED BY THE NORMAL CURVES OF ORDER n IN A SPACE OF n DIMENSIONS.

BY PROFESSOR J. W. YOUNG.

(Read before the American Mathematical Society, February 29, 1908.)

A NORMAL curve C_n of order n in a space S_n of n dimensions is transformed into itself by ∞^3 collineations

$$z_i' = \sum_{k=0}^{n} a_{ik} z_k \qquad (i = 0, 1, \cdots, n)$$

in S_n. Each of these collineations subjects the parameter $\zeta_1 : \zeta_2$ of C_n to a linear substitution $\zeta_1' = \alpha\zeta_1 + \beta\zeta_2$, $\zeta_2' = \gamma\zeta_1 + \delta\zeta_2$. The group of collineations obtained by restricting the coefficients a_{ik} to rational integral values with determinant $|a_{ik}| = 1$ is properly discontinuous, and the corresponding group Γ_n of linear fractional substitutions

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \qquad (\alpha\delta - \beta\gamma = 1), \tag{1}$$

on the parameter $\zeta = \zeta_1 : \zeta_2$ is hence likewise properly discontinuous. This latter group Γ_n we call the discontinuous ζ-group defined by C_n.

The arithmetic definition of discontinuous groups of substitutions (1) forms one of the fundamental problems in the theory of automorphic functions. The above is an outline of one of the few effective methods that have been suggested for this purpose;* but on account of its complexity little has been done with it for values of $n > 2$.† General results applying to any value of n are almost totally lacking. The case $n = 2$. however, has been exhaustively treated on the arithmetic side by Fricke,‡ and has yielded results of great importance. Funda-

* Fricke, Chicago Congress Papers, p. 85, and J. W. Young, "On a class of discontinuous ζ-groups, etc.," *Rendiconti del Circolo matematico di Palermo*, vol. 23 (1907), p. 97.

† The case $n = 4$ has been partially treated by the author, loc. cit.

‡ Fricke, "Ueber indefinite Formen mit drei und vier Variabeln," *Göttinger Nachrichten*, Dec. 13, 1883. Cf. also Fricke-Klein, "Theorie der automorphen Functionen" (Leipzig, 1897), vol. 1, p. 502 ff.

mental in Fricke's treatment of the groups Γ_2 is an invariant $J = (\alpha + \delta)^2 - 1$, from the consideration of which follows that any group Γ_2 contains no elliptic substitutions of periods other than 2, 3, 4, and 6.* It is the object of this paper to calculate the corresponding invariant for the groups Γ_n in general, and to derive the corresponding general theorem concerning the existence of elliptic substitutions in any Γ_n.

The equations of any normal curve C_n can by linear transformation be reduced to the form $z_i = \zeta_1^{n-i}\zeta_2^i$ $(i = 0, 1, \cdots, n)$. We proceed first to derive the most general collineation in S_n which leaves this curve invariant. It is readily obtained by subjecting the parameter $\zeta_1 : \zeta_2$ of the curve to the substitution

$$(1') \qquad \zeta_1' = \alpha\zeta_1 + \beta\zeta_2, \quad \zeta_2' = \gamma\zeta_1 + \delta\zeta_2 \quad (\alpha\delta - \beta\gamma = 1).$$

This gives

$$z_i' = (\alpha\zeta_1 + \beta\zeta_2)^{n-i}(\gamma\zeta_1 + \delta\zeta_2)^i \quad (i = 0, 1, \cdots, n).$$

If the right-hand member be expanded and the products $\zeta_1^\lambda\zeta_2^\mu$ replaced by their equivalent z_j's, the desired collineation is obtained in the form

$$(2) \quad z_i' = \sum_{k=0}^{n} a_{ik}z_k = \sum_{j=0}^{n-i}\sum_{j'=0}^{i}\binom{n-i}{j}\binom{i}{j'}\alpha^{n-i-j}\beta^j\gamma^{i-j'}\delta^{j'}z_{j+j'}.$$

The collineation which transforms any C_n whatever into itself is then obtained from this by linear transformation, since all C_n's are projectively equivalent. However, it is well-known that under every such transformation the sum Σa_{ii} of the leading diagonal elements of the matrix (a_{ik}) is an absolute invariant. This invariant, which we will denote by J, turns out to be a polynomial in $\alpha + \delta$ with integral coefficients. It may be calculated as follows:

From (2) we have at once, since a_{ii} is the coefficient of z_i,

$$J = \sum_{i=0}^{n} a_{ii} = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n-i}{j}\binom{i}{i-j}\alpha^{n-i-j}\delta^{i-j}\beta^j\gamma^j.$$

If in this expression we replace $\beta\gamma$ by $\alpha\delta - 1$, we obtain

$$J = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n-i}{j}\binom{i}{i-j}\alpha^{n-i-j}\delta^{i-j}\sum_{\nu=0}^{j}(-1)^\nu\binom{j}{\nu}\alpha^{j-\nu}\delta^{j-\nu}$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{\nu=0}^{j}(-1)^\nu\binom{n-i}{j}\binom{i}{i-j}\binom{j}{\nu}\alpha^{n-i-\nu}\delta^{i-\nu}.$$

* Fricke-Klein, loc. cit., p. 517.

Now we readily verify the following relations between binomial coefficients:

$$(3) \qquad \binom{p}{j}\binom{j}{q} = \binom{p}{q}\binom{p-q}{j-q} = \binom{p}{q}\binom{p-q}{p-j}.$$

The first of these gives

$$\binom{n-i}{j}\binom{j}{\nu} = \binom{n-i}{\nu}\binom{n-i-\nu}{j-\nu},$$

whence substituting we have

$$J = \sum_i \sum_j \sum_\nu (-1)^\nu \binom{n-i}{\nu}\binom{i}{i-j}\binom{n-i-\nu}{j-\nu} \alpha^{n-i-\nu}\delta^{i-\nu}.$$

Here and in all that follows, the summations are to be extended over all possible combinations of the symbols involved. The order in which the summations are carried out is clearly immaterial. In the last expression for J we replace $\binom{i}{i-j}$ by $\binom{i}{j}$ and j by $\nu + j'$ and sum first with respect to j', making use of the formula*

$$\sum_{s=0} \binom{p}{n+s}\binom{q}{s} = \binom{p+q}{n+q}.$$

We thus obtain

$$J = \sum_i \sum_\nu \sum_{j'} (-1)^\nu \binom{n-i}{\nu}\binom{i}{\nu+j'}\binom{n-i-\nu}{j'} \alpha^{n-i-\nu}\delta^{i-\nu}$$

$$= \sum_i \sum_\nu (-1)^\nu \binom{n-i}{\nu}\binom{n-\nu}{n-i} \alpha^{n-i-\nu}\delta^{i-\nu}.$$

By the second part of (3), this may be written

$$J = \sum_\nu \sum_i (-1)^\nu \binom{n-\nu}{\nu}\binom{n-2\nu}{i-\nu} \alpha^{n-i-\nu}\delta^{i-\nu},$$

which is clearly equivalent to

$$(4) \qquad J = \sum_{\nu=0} (-1)^\nu \binom{n-\nu}{\nu} (\alpha+\delta)^{n-2\nu}.$$

This is the desired invariant. For the value $n = 2$ it evidently reduces to the expression already referred to for this case. The result obtained may be stated as follows:

* Netto, Lehrbuch der Combinatorik, Leipzig, 1901, p. 250.

THEOREM 1. *If the parameter ζ of any normal curve of order n in a space S_n of n dimensions is transformed by a substitution* (1), *the sum J of the elements in the leading diagonal of the matrix of the corresponding collineation in S_n is independent of the curve and is given by* (4).

This yields at once a condition on the periods of the elliptic substitutions in any discontinuous group Γ_n obtained by restricting the coefficients in the collineations to rational integral values. For every substitution (1) of any such group Γ_n, *the number J must be an integer.* Now, if ω be the period of any elliptic substitution (1) in a Γ_n we have*

$$\pm(\alpha+\delta)=2\cos\frac{\pi}{\omega}. \tag{5}$$

If this expression be substituted in (4), remembering that J is an integer, we obtain an equation which ω must satisfy. This may be put into a more convenient form by placing $x=e^{\pi i/\omega}$, and hence placing $(\alpha+\delta)=x+x^{-1}$.† We obtain readily

$$J=\sum_{\nu}\sum_{\mu}(-1)^{\nu}\binom{n-\nu}{\nu}\binom{n-2\nu}{\mu}x^{n-2\nu-2\mu},$$

or placing $\nu+\mu=p$, we obtain the coefficient of x^{n-2p} in the form

$$\sum_{\nu}(-1)^{\nu}\binom{n-\nu}{\nu}\binom{n-2\nu}{p-\nu}.$$

By applying the first relation (3) in the opposite direction, we obtain

$$\binom{n-\nu}{\nu}\binom{n-2\nu}{p-\nu}=\binom{n-\nu}{p}\binom{p}{\nu}.$$

But by a well-known relation we have‡

$$\sum_{\nu}(-1)^{\nu}\binom{n-\nu}{p}\binom{p}{\nu}=\binom{n-p}{n-p}=1.$$

We have therefore the relation

$$J=x^n+x^{n-2}+x^{n-4}+\cdots+x^{-n+2}+x^{-n},$$

* Fricke-Klein, loc. cit., p. 517.

† The signs of α, δ can always be so chosen that the upper sign in (5) holds.

‡ Netto, loc. cit., p. 252.

or multiplying by x^n,

$$x^{2n} + x^{2n-2} + x^{2n-4} + \cdots + x^2 + 1 = Jx^n.$$

Since $x^2 \neq 1$, this may also be written

$$x^{2n+2} - 1 = Jx^n(x^2 - 1). \tag{6}$$

A still more convenient form of this equation, for some purposes, may be obtained by placing $x = e^{\pi i/\omega} = \cos\omega + i\sin\omega$ and equating real and imaginary parts. By means of elementary transformations (6) is then readily shown to be equivalent to the equation

$$\sin\frac{(n+1)\pi}{\omega} = J\sin\frac{\pi}{\omega}. \tag{7}$$

These results may be stated as follows:

THEOREM 2. *In any discontinuous ζ-group defined by a normal curve C_n in a space S_n the period ω of any elliptic substitution must satisfy an equation* (7), *where J is an integer, or* (6), *where $x = e^{\pi i/\omega}$ and J is an integer.*

The numbers 2, 3 are possible periods for every value of n, as may be easily verified from the above condition. This follows also at once from the fact that the elliptic modular group, which contains substitutions of periods 2 and 3, is a Γ_n for every value of n. In fact it is the group defined by the canonical C_n which we used above to obtain the desired invariant.§ It has already been pointed out that the only other possible periods in the case $n = 2$ are 4 and 6. As examples of the application of the last theorem we give in the table below the possible periods for the values of n from 3 to 6 inclusive, calculated from (6):

Possible periods.

$n = 3$ 2, 3, 4, 5;
$n = 4$ 2, 3, 4, 5, 6;
$n = 5$ 2, 3, 5, 6, 7;
$n = 6$ 2, 3, 4, 6, 7, 8.

Moreover, by placing $J = 0$ and ± 1, it is readily seen that every factor of n, $n+1$, and $n+2$ is a possible period in any Γ_n.

PRINCETON UNIVERSITY,
February, 1908.

§ Cf. Young, loc. cit., p. 99, footnote.

ON CERTAIN CONSTANTS ANALOGOUS TO FOURIER'S CONSTANTS.

BY MR. CHARLES N. MOORE.

(Read before the American Mathematical Society, April 25, 1908.)

In the course of an article which appeared recently in the *Rendiconti del Circolo Matematico di Palermo*,* Landau has reproduced two proofs of the following theorem:

A. If the function $\psi(x)$ *is continuous in the interval* $(0 \leqq x \leqq 1)$ *and if*

$$\int_0^1 x^\nu \psi(x) dx = 0 \qquad (\nu = 0, 1, 2, \cdots), \tag{1}$$

then

$$\psi(x) = 0 \qquad (0 \leqq x \leqq 1).$$

The proofs that Landau gives in detail are due to Lerch and Stieltjes. In addition he cites a second proof due to Stieltjes and a proof due to Phragmen.

As far as I am able to learn, no one seems to have mentioned the fact that this theorem, of which so many proofs have been given, is essentially equivalent to a theorem due to Hurwitz† which may be stated as follows:

B. If in the interval $(0 \leqq x \leqq 2\pi)$ *the function* $f(x)$ *is finite and integrable and if all of its Fourier's constants are zero, then* $f(x)$ *is zero at every point of the interval at which it is continuous.*

Theorem (A) may be deduced from (B) as follows:

It is obvious that if $\psi(x)$ is finite and integrable in the interval $(0 \leqq x \leqq 1)$ and if condition (1) is fulfilled, then the function

$$f(y) = \psi(y/2\pi)$$

satisfies all the conditions of Hurwitz's theorem. For

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* Vol. 25 (1908), p. 1.

† Cf. *Mathematische Annalen*, vol. 57 (1903), p. 440. Cf. also Bonnet, *Mémoires de l' Académie de Belgique*, vol. 23 (1850), p. 11.

$$\frac{1}{\pi}\int_0^{2\pi} f(y)\cos nydy = 2\int_0^1 \psi(x)\cos 2n\pi x dx$$

$$= 2\left[\int_0^1 \psi(x)dx - \frac{4n^2\pi^2}{2!}\int_0^1 x^2\psi(x)dx + \cdots\right] = 0^*,$$

and similarly

$$\frac{1}{\pi}\int_0^{2\pi} f(y)\sin nydy = 0.$$

Consequently $f(y)$ is zero at every point of the interval $(0 \leqq y \leqq 2\pi)$ at which it is continuous, and hence $\psi(x)$ is zero at every point of the interval $(0 \leqq x \leqq 1)$ at which it is continuous.

Thus Hurwitz's theorem gives us an immediate proof of theorem (A) in the more general case in which $\psi(x)$ is merely finite and integrable. However, as Professor Bôcher has pointed out,† Hurwitz's theorem holds even when $f(x)$ becomes infinite at a finite number of points, provided

$$\int_0^{2\pi} |f(x)|\,dx$$

converges. Therefore theorem (A) holds for the case in which

$$\int_0^1 |\psi(x)|\,dx$$

is convergent.‡

I will now give a rather simple proof of theorem (A) for a still more general case and will then obtain theorem (B) under equally general conditions, as a consequence of this theorem. As far as I know, neither theorem has been proved before with the same degree of generality.

I will begin by proving two lemmas.

LEMMA 1. *If $\psi(x)$ is continuous in the interval $(0 \leqq a \leqq x \leqq b)$ and if*

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* We have a right to multiply the series for the cosine by a function that is finite and integrable and to integrate it term by term since the series is uniformly convergent throughout any finite interval.

† Cf. *Annals of Mathematics*, vol. 7 (1906), p. 101.

‡ This is in view of the fact that we have a right to multiply the series for the cosine by a function that is absolutely integrable and to integrate term by term.

$$\int_a^b x^{\gamma+n}\psi(x)dx = 0 \quad (\gamma \geqq 0;\ n = 0, 1, 2, \cdots), \tag{2}$$

then

$$\psi(x) = 0 \qquad (a \leqq x \leqq b).$$

Since $\psi(x)$ is continuous, we know from a theorem due to Weierstrass * that we can develop it in a uniformly convergent series of polynomials in x so that we have

$$\psi(x) = P_0(x) + P_1(x) + P_2(x) + \cdots. \tag{3}$$

We now multiply both sides of equation (3) by $x^\gamma\psi(x)$, and integrate from a to b. Since the series on the right hand side of (3) is uniformly convergent, we have a right to integrate term by term and hence in view of condition (2) the whole right hand side vanishes, so that we get

$$\int_a^b x^\gamma[\psi(x)]^2 dx = 0.$$

Consequently, since $\psi(x)$ is continuous, it must be zero at every point of the interval $(a \leqq x \leqq b)$ and the lemma is proved.

By a change of variable this lemma can be thrown into the slightly more general form :

LEMMA 2. *If $\psi(x)$ is continuous in the interval $(0 \leqq a \leqq x \leqq b)$ and if*

$$\int_a^b x^{\gamma+n\alpha}\psi(x)dx = 0 \quad (\gamma \geqq 0,\ \alpha > 0;\ n = 0, 1, 2, \cdots), \tag{4}$$

then

$$\psi(x) = 0 \qquad (a \leqq x \leqq b).$$

We are now in a position to prove the following two theorems :

THEOREM I. *If in the interval $(0 \leqq a \leqq x \leqq b)$ $\psi(x)$ is finite save for a finite number of points, and is integrable, and if furthermore*

$$\int_a^b x^{\gamma+n\alpha}\psi(x)dx = 0 \dagger \quad (\gamma \geqq 0,\ \alpha > 0;\ n = 0, 1, 2, \cdots), \tag{5}$$

* Cf. Picard, Traité d'Analyse, vol. 1, 2d ed., p. 279. Lerch's proof referred to above is based on a slightly different form of this theorem. The proof as I have given it, however, is considerably briefer.

† Condition (5) is not essentially more general than condition (1), since it involves merely a change of notation. The real generalization obtained in Theorem I is in the removal of restrictions upon $\psi(x)$.

then $\psi(x)$ is zero at every point of the interval $(a \leqq x \leqq b)$ at which it is continuous.

Let

$$\int_a^x x^\gamma \psi(x)dx = \chi(x). \tag{6}$$

Then $\chi(x)$ is continuous in the interval $(a \leqq x \leqq b)$ and in view of (6) and (5)

$$\chi(a) = 0, \quad \chi(b) = 0. \tag{7}$$

Let us take the integer m so great that $m\alpha > 1$. Then for $n = m + k \geqq m$ we have from an integration by parts

$$\int_a^b x^{\gamma+n\alpha}\dot{\chi}(x)dx = [x^{n\alpha}\chi(x)]_a^b - \int_a^b x^{m\alpha-1+k\alpha}\chi(x)dx \ (k=0,1,2,\cdots)$$

or in view of (5) and (7)

$$\int_a^b x^{m\alpha-1+k\alpha}\chi(x)dx = 0 \quad (k = 0, 1, 2, \cdots)$$

and hence from Lemma 2

$$\chi(x) = \int_a^x x^\gamma \psi(x)dx = 0 \qquad (a \leqq x \leqq b). \tag{8}$$

Differentiating (8), we see that $x^\gamma\psi(x)$, and consequently $\psi(x)$, is zero at every point of the interval $(a \leqq x \leqq b)$ at which it is continuous.*

THEOREM II. *If in the interval $(0 \leqq a \leqq x \leqq b)$ $\psi(x)$ is finite save for a finite number of points, and is integrable, and if furthermore*

$$\int_a^b \psi(x) \cos nx dx = 0 \quad (n = 0, 1, 2, \cdots),$$
$$\int_a^b \psi(x) \sin nx dx = 0 \qquad (n = 1, 2, \cdots), \tag{9}$$

then $\psi(x)$ is zero at every point of the interval $(a \leqq x \leqq b)$ at which it is continuous.

* If $a = 0$ and $\gamma > 0$ it does not necessarily follow that, when $x^\gamma\psi(x)$ is zero for $x = 0$, $\psi(x)$ is zero also. However, when (8) is fulfilled for $a = 0$, it is easily seen from the integrability of $\psi(x)$ that it must be zero for $x = 0$ if it is continuous there.

We know that in any fixed interval x^n can be developed in a Fourier's series that is uniformly convergent throughout the interval and which when differentiated term by term will yield a series that is uniformly convergent throughout the same interval and which represents there the derivative of x^n.* Hence we have

$$x^n = a_0 + a_1 \cos x + b_1 \sin x + \cdots \quad (a \leqq x \leqq b), \tag{10}$$

$$nx^{n-1} = -a_1 \sin x + b_2 \cos x - \cdots \quad (a \leqq x \leqq b). \tag{11}$$

In view of (10) and the hypothesis that $\psi(x)$ is integrable in the interval $a \leqq x \leqq b$, we have

$$\begin{aligned} b^n \int_a^b \psi(x)dx = a_0 \int_a^b \psi(x)dx + a_1 \cos b \int_a^b \psi(x)dx \\ + b_1 \sin b \int_a^b \psi(x)dx + \cdots. \end{aligned} \tag{12}$$

Now let

$$\int_a^x \psi(x)dx = \chi(x). \tag{13}$$

Then $\chi(x)$ is continuous in the interval $(a \leqq x \leqq b)$, and since the series on the right hand side of (11) is uniformly convergent, we have

$$\begin{aligned} -n \int_a^b x^{n-1}\chi(x)dx = a_1 \int_a^b \chi(x) \sin x dx \\ - b_1 \int_a^b \chi(x) \cos x dx + \cdots. \end{aligned} \tag{14}$$

Adding (12) and (14), we get

$$\begin{aligned} b^n \int_a^b \psi(x)dx - n \int_a^b x^{n-1}\chi(x)dx \\ = a_0 \int_a^b \psi(x)dx + a_1 \left[\cos b \int_a^b \psi(x)dx + \int_a^b \chi(x) \sin x dx \right] \\ + b_1 \left[\sin b \int_a^b \psi(x)dx - \int_a^b \chi(x) \cos x dx \right] + \cdots \end{aligned}$$

or

* For sufficient conditions that a function may be developed in a uniformly convergent Fourier's series and that a uniformly convergent Fourier's series representing its derivative may be obtained by differentiating that series term by term see Professor Bôcher's article referred to above.

$$\int_a^b x^n \psi(x)dx = a_0 \int_a^b \psi(x)dx + a_1 \int_a^b \psi(x) \cos x dx$$

$$+ b_1 \int_a^b \psi(x) \sin x dx + \cdots = 0 \quad (n = 1, 2, \cdots).$$

Hence $\psi(x)$ satisfies the conditions of Theorem I, and therefore it is zero at every point at which it is continuous.

The method of proof used in Theorem II may be applied in the case of developments in terms of any other normal functions, such as Bessel functions, Legendre's polynomials, etc., whenever we know that x^n can be developed in a convergent series of such functions which, when differentiated term by term, will yield a uniformly convergent series that represents the derivative of x^n.* The method will enable us to show in such cases that if the coefficients of the development corresponding to any function which we know to be finite, save for a finite number of points, and integrable, are all zero, the function is zero at every point at which it is continuous.

NOTE ON THE SECOND VARIATION IN AN ISOPERIMETRIC PROBLEM.

BY DR. ELIJAH SWIFT.

(Read before the American Mathematical Society, April 25, 1908.)

SUPPOSE we have before us the simplest type of isoperimetric problem, namely to determine x and y as functions of a parameter t, so that the definite integral

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

shall be a minimum, while another definite integral

$$K = \int_{t_0}^{t_1} G(x, y, x', y')dt$$

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* The existence of such developments can be proved for some of these cases by means of some theorems discussed by Stekloff. Cf. *Mémoires de l'Académie de St. Pétersbourg*, ser. 8, vol. 15 (1904).

takes a given value l, the functions $x(t)$ and $y(t)$ being subject to the condition that they take given values x_0, y_0 and x_1, y_1 for $t = t_0$ and $t = t_1$ respectively.

The object of this note is to point out a way of obtaining the second variation of J that seems to me clearer than that found in most text-books,* and which supplies the reasons for the usual derivation.

If we have a solution of the problem, $x = x(t)$, $y = y(t)$, we vary the integral J by replacing x and y by $x + \xi$ and $y + \eta$ respectively, where for all variations that leave K unchanged ξ and η may be expressed as power series in a quantity ϵ_1, if for convenience sake we assume that K and F are analytic, and taking ϵ_1 as principal infinitesimal, are in general of the first order.† We thus obtain a new integral $J + \Delta J$. The first variation of J consists of all terms of ΔJ which are of the first order; the second variation, of all those of the second order, and so on.

To obtain the second variation Bolza proceeds as follows:‡ Since for all admissible variations K must remain equal to l, we must have $\Delta K = 0$. ΔJ is therefore unchanged if we add any multiple of ΔK to it, in particular $\Delta J = \Delta J + \lambda_0 \Delta K$, where λ_0 is the isoperimetric constant.§ Developing this expression and taking the terms of the second degree in ξ, η and their derivatives we have the usual expression for the second variation $\Delta^2 J$.

We naturally ask: Why add ΔK at all to ΔJ? And why add exactly $\lambda_0 \Delta K$? Moreover if $\Delta K = 0$, how does it happen that adding it to ΔJ makes any difference at all in the second variation? In considering a special isoperimetric problem ‖ I was led to attempt to answer these questions.

The total variation of J is

$$\Delta J = \int_{t_0}^{t_1} \left[\left(\frac{\partial F}{\partial x}\xi + \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial x'}\xi' + \frac{\partial F}{\partial y'}\eta' \right) + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}\xi^2 + \cdots + \frac{\partial^2 F}{\partial y'^2}\eta'^2 \right) + \cdots \right] dt.$$

* E. g., Bolza, "Lectures on the Calculus of Variations," p. 216.
† Bolza, pp. 206 ff.
‡ Bolza, loc. cit.
§ Bolza, pp. 208, 209.
‖ Dissertation, Göttingen, 1907, pp. 16 ff.

To find the second variation, we must take all terms that are of the second order. *But these come, not merely from the terms of the second degree in ξ, η and their derivatives, but also from those of the first, since ξ and η are power series in ϵ_1 and involve other powers of ϵ_1 besides the first.* We must take then all terms of the second order in ΔJ. Then we have $\Delta^2 J =$ all terms of the second order in

$$\int_{t_0}^{t_1}\left[\left(\frac{\partial F}{\partial x}\xi+\cdots+\frac{\partial F}{\partial y'}\eta'\right)+\frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}\xi^2+\cdots+\frac{\partial^2 F}{\partial y'^2}\eta'^2\right)\right]dt.$$

The question comes up — How pick out the terms of the second order in the first term in ΔJ and so obtain *an explicit expression* for the second variation?

We can do this by using the fact that since $\Delta K = 0$, all the terms of the second order in ΔK must vanish, and moreover the first variation of ΔJ is zero.* The two corresponding equations are $\Delta^2 K =$ all terms of the second order in

$$\int_{t_0}^{t_1}\left[\left(\frac{\partial G}{\partial x}\xi+\cdots+\frac{\partial G}{\partial y'}\eta'\right)+\frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}\xi^2+\cdots+\frac{\partial^2 G}{\partial y'^2}\eta'^2\right)\right]dt=0$$

and

$$\int_{t_0}^{t_1}\left[\left(\frac{\partial F}{\partial x}\xi+\cdots+\frac{\partial F}{\partial y'}\eta'\right)+\lambda_0\left(\frac{\partial G}{\partial x}\xi+\cdots+\frac{\partial G}{\partial y'}\eta'\right)\right]dt=0.$$

It is at once evident that we can eliminate the terms of the first degree in ΔJ by adding to it $\lambda_0 \Delta K$, and so obtain an explicit expression for the second variation. If we call $F+\lambda_0 G = H$, we have $\Delta^2 J = \Delta^2 J + \lambda_0 \Delta^2 K =$ all terms of the second order in

$$\frac{1}{2}\int_{t_0}^{t_1}\left(\frac{\partial^2 H}{\partial x^2}\xi^2+\cdots+\frac{\partial^2 H}{\partial y'^2}\eta'^2\right)dt,$$

the form in which the second variation is usually written.

The advantage of adding $\lambda_0 \Delta K$ to ΔJ is then that by so doing we obtain the terms of the second order in the isoperimetric problem explicitly from H in the same way as in the ordinary problem from F.

PRINCETON, N. J.,
February, 1908.

* Bolza, p. 209.

NOTE ON A CERTAIN EQUATION INVOLVING THE FUNCTION $E(x)$.

BY PROFESSOR R. D. CARMICHAEL.

RECENTLY J. V. Pexider has studied the equation*

$$E\left(\frac{n+\alpha}{x}\right) - E\left(\frac{n+\alpha}{x+1}\right) = d, \tag{1}$$

where $E(s)$ is the greatest integer $\leqq s$, and where α is zero or a positive quantity less than 1, n is a known positive integer, d is zero or a known positive integer, and x is an unknown positive integer. M. Pexider confines himself chiefly to the case in which $d = 0$ and x is less than n. He finds the values of x which satisfy the equation subject to these restrictions.

In the present note it is proposed to exhibit a simple working method by which the roots can be found in any case. When $d > 0$, x is always less than n, except that x may equal n when $d = 1$. In what follows x is taken always less than n.

If $n + \alpha$ is divided by an integer i, giving the quotient $q + \beta$ where β is zero or a positive quantity less than 1; then if n is also divided by i, the quotient will evidently be $q + y$ where y is zero or a positive quantity less than 1. Hence

$$E\left(\frac{n+\alpha}{i}\right) = E\left(\frac{n}{i}\right).$$

Therefore, the equation

$$E\left(\frac{n}{x}\right) - E\left(\frac{n}{x+1}\right) = d \tag{2}$$

has the same roots as (1). We may then confine ourselves to the solution of the latter equation as being somewhat the simpler of the two.

Represent n in the form

$$n = ax + c \qquad (c < x,\ a \neq 0). \tag{3}$$

* *Rendiconti del Circolo Matem. di Palermo*, vol. 24, no. 1, pp. 46–64. For convenience I write the equation in a form somewhat different from that of M. Pexider.

Then

$$E\left(\frac{n}{x}\right) \equiv E\left(\frac{ax+c}{x}\right) = a;$$

and therefore

$$E\left(\frac{n}{x+1}\right) \equiv E\left(\frac{ax+c}{x+1}\right) = a-d \tag{4}$$

is the necessary and sufficient condition that x is a root of (2). Now (4) will be satisfied if and only if

$$ax + c - (x+1)(a-d) < x+1 \quad \text{and} \quad \geqq 0.$$

This readily reduces to

$$a - c > (d-1)(x+1) \quad \text{and} \quad \leqq d(x+1). \tag{5}$$

From these results we have the following working method for finding the solutions of (2) and hence of (1):

Write n in every possible way in the form

$$n = ax + c \qquad (c < x,\ a \neq 0) \tag{3}$$

and examine whether both inequalities

$$a - c > (d-1)(x+1) \quad \text{and} \quad \leqq d(x+1) \tag{5}$$

are fulfilled at the same time; if so, x is a root of the equation; otherwise, it is not such a root.

Applying this to the case when $d = 0$, we have to determine x subject to the conditions

$$n = ax + c, \quad (c < x,\ a \neq 0,\ c - a < x + 1 \quad \text{and} \quad \geqq 0).$$

M. Pexider (l. c., page 57) shows that the number of such roots (x being less than n) is

$$A(n) = E(n) - E(\sqrt{n}) - E\left(\frac{n}{E(\sqrt{n})+1}\right).$$

He exhibits also a better method than the above by which the roots may be determined in the present case.

If $d = 1$, the roots of our equation are to be found from

$$n = ax + c, \quad (c < x,\ a \neq 0,\ a - c > 0 \quad \text{and} \quad \leqq x + 1).$$

This will give all the roots of (1) and (2) except $x = n$ which has

been definitely excluded from this result by previous assumption that $x < n$.

For other values of d the expressions in (5) do not simplify, and we have to determine x from the general conditions

$$n = ax + c,\ c < x,\ (a \neq 0,\ a - c > (d-1)(x+1) \text{ and } \leqq d(x+1)).$$

It may be verified by actual trial that the solutions of (1) and (2) may in this way be effected with a degree of readiness which will make the method serviceable in a large number of cases.

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THE INNER FORCE OF A MOVING ELECTRON.

BY DR. F. R. SHARPE.

(Read before the American Mathematical Society, February 29, 1908.)

1. *Introduction.* — Sommerfeld * has given a general method of determining the force which an electron exerts on itself when its motion is known. Schott † has applied the retarded potential to the same problem. In the present paper the known vector expression ‡ for the force with which a moving point charge acts on another point charge is used to determine the action between any two elements of the electron. A double integration over the volume of the electron gives the inner force of the electron. The Abraham-Sommerfeld expressions for the longitudinal mass and Abraham's value for the transverse mass are thus very simply determined and the limitations on the solutions are made manifest.

2. *The Force Between Two Moving Point Charges.* — The electron is assumed to be a uniformly charged solid sphere which is moving in a straight line without rotation. Take the x-axis in the direction of motion. Measure the time from the instant at which the force is to be determined and choose the velocity of light as the unit of velocity. Let (x_0, y, z) be the coordinates of the point charge de_2 relative to the point charge de_1. The distance moved by the electron in time t is

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* *Göttinger Nachrichten*, 1904.
† *Ann. der. Physik*, 1908, No. 1.
‡ Abraham : Theorie der Elektrizität, vol. II, p. 98.

$$v_0 t + \tfrac{1}{2}\dot{v}_0 t^2 + \tfrac{1}{6}\ddot{v}_0 t^3 + \cdots,$$

where v_0 is its velocity at time 0. The point charge de_1 will therefore act on de_2 at the time $-\tau$ which is determined so as to satisfy

$$\tau^2 = y^2 + z^2 + (x_0 + v_0\tau - \tfrac{1}{2}\dot{v}_0\tau^2 + \cdots)^2. \tag{1}$$

The velocity of de_1 at this time is

$$v = v_0 - \dot{v}_0\tau + \tfrac{1}{2}\ddot{v}_0\tau^2 - \cdots, \tag{2}$$

and its acceleration is

$$\dot{v} = \dot{v}_0 - \ddot{v}_0\tau + \cdots. \tag{3}$$

If the velocity of de_1 had remained constant and equal to v from the time $-\tau$, its coordinates at time 0 would have been (x, y, z), where

$$x = x_0 + v_0\tau - \tfrac{1}{2}\ddot{v}_0\tau^2 + \cdots - \tau(v_0 - \dot{v}_0\tau + \cdots)$$

or

$$x = x_0 + \tfrac{1}{2}\dot{v}_0\tau^2 + \cdots. \tag{4}$$

The x component of the force of de_1 on de_2 can be reduced from its vector form to

$$\frac{x(1-v^2) - \dot{v}(y^2+z^2)}{\{x^2 + (1-v^2)(y^2+z^2)\}^{\frac{3}{2}}}\, de_1 de_2. \tag{5}$$

The effect of de_2 on de_1 is obtained by changing the sign of x_0.

3. *Uniform Motion.*—In this case v is v_0 and x is x_0, so that (5) becomes

$$\frac{x_0(1-v_0^2)}{\{x_0^2 + (1-v_0^2)(y^2+z^2)\}^{\frac{3}{2}}}\, de_1 de_2.$$

Hence the mutual actions of de_1 and de_2 are equal and opposite and the inner force is zero.

4. *Slow Uniform Acceleration.*—Here $\dot{v}$ is $\dot{v}_0$, and the unbalanced part of the force of de_1 on de_2 is

$$-\frac{\dot{v}_0(y^2+z^2)de_1 \cdot de_2}{\{x_0^2 + (1-v_0^2)(y^2+z^2)\}^{\frac{3}{2}}},$$

which may be written

$$-\frac{\dot{v}_0\partial}{v_0\partial v_0}\left[\frac{de_1 de_2}{\{x_0^2+(1-v_0^2)(y^2+z^2)\}^{\frac{1}{2}}}\right].$$

The inner force of the electron is twice the sum of this expression for every pair of elements of the electron. Let the coordinates of de_1 and de_2 referred to the center of the electron be (x_1, y_1, z_1) and (x_2, y_2, z_2). Then

$$x_0 = x_2 - x_1, \quad y = y_2 - y_1, \quad \text{and} \quad z = z_2 - z_1.$$

If we now make the transformation

$$x = \sqrt{1-v_0^2}\cdot x', \quad \rho = \frac{\rho'}{\sqrt{1-v_0^2}},$$

where ρ is the volume density of the charge e of the electron, the charge is unaltered, but the spherical electron of radius a is transformed into the prolate spheroid

$$x'^2(1-v_0^2) + y^2 + z^2 = a^2.$$

The expression which we have to integrate is therefore $2/\sqrt{(1-v_0^2)}$ of the self potential of this spheroid and has the value *

$$\frac{6}{5}\cdot\frac{e^2}{2a\dot{v}_0}\log\left(\frac{1+v_0}{1-v_0}\right).$$

Hence the inner force of the electron is

$$-\frac{6}{5}\cdot\frac{e^2\dot{v}_0}{2a}\cdot\frac{\partial}{v_0\partial v_0}\left\{\frac{1}{v_0}\log\left(\frac{1+v_0}{1-v_0}\right)\right\},$$

that is,

$$-\dot{v}_0\cdot\frac{6}{5}\frac{e^2}{2av_0^2}\left(-\frac{1}{v_0}\log\frac{1+v_0}{1-v_0}+\frac{2}{1-v_0^2}\right).$$

The coefficient of $-\dot{v}_0$ is Abraham's expression for the longitudinal mass in the case of quasi-stationary motion.

5. *Variable Acceleration.* — In the previous case the changes in the velocity and acceleration were neglected. If we retain the first power of τ, we have from (2) and (3)

$$v = v_0 - \dot{v}_0\tau, \quad \dot{v} = \dot{v}_0 - \ddot{v}_0\tau,$$

* Abraham, *loc. cit.*, p. 179.

and from (1)

$$\tau = \frac{v_0 x + \sqrt{x_0^2 + (1 - v_0^2)(y^2 + z^2)}}{1 - v_0^2}.$$

Substituting in (5), we find the additional unbalanced terms

$$\frac{\ddot{v}_0 (y^2 + z^2) de_1 de_2}{(1 - v_0^2)\{x_0^2 + (1 - v_0^2)(y^2 + z^2\}} + \frac{3\dot{v}_0^2 v_0 (y^2 + z^2)^2 de_1 de_2}{(1 - v_0^2)\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}^2}.$$

Denoting by θ the angle which the line joining de_1 to de_2 makes with the x axis, we have

$$\frac{\ddot{v}_0}{1 - v_0^2} \cdot \frac{\sin^2 \theta de_1 de_2}{1 - v_0^2 \sin^2 \theta} + \frac{3\dot{v}_0^2}{2(1 - v_0^2)} \frac{\partial}{\partial v_0} \left(\frac{\sin^2 \theta de_1 de_2}{1 - v_0^2 \sin^2 \theta} \right).$$

Consider

$$\int\int \frac{\sin^2 \theta de_1 de_2}{1 - v_0^2 \sin^2 \theta};$$

since it is independent of the length of the line joining de_1 to de_2, its value is

$$\frac{e^2}{4\pi} \int \frac{\sin^2 \theta}{1 - v_0^2 \sin^2 \theta} dw,$$

where dw is the element $2\pi \sin \theta d\theta$ of the complete solid angle 4π. Integrating from 0 to π, we find the value of the additional terms to be

$$\frac{\ddot{v}_0 e^2}{1 - v_0^2} \left(-\frac{1}{v_0^2} + \frac{1}{v_0^3 \sqrt{1 - v_0^2}} \tan^{-1} \frac{v_0}{\sqrt{1 - v_0^2}} \right)$$
$$+ \frac{3\dot{v}_0^2 e^2}{2(1 - v_0^2)} \frac{\partial}{\partial v_0} \left(-\frac{1}{v_0^2} + \frac{1}{v_0^3 \sqrt{1 - v_0^2}} \tan^{-1} \frac{v_0}{\sqrt{1 - v_0^2}} \right).$$

This result differs somewhat from the expression found by Schott and Abraham.

6. *Transverse Acceleration.*—When the velocity is constant but there is a small transverse acceleration f, we may choose the y axis in the direction of f. The vector expression for the y component of the unbalanced force may be reduced to

$$-f \frac{x_0^2 + (1 - v_0^2) z^2}{\{x_0^2 + (1 - v_0^2)(y^2 + z^2)\}^{\frac{3}{2}}} de_1 de_2,$$

that is,

$$-fde_1de_2\left[\frac{1}{\{x_0^2+(1-v_0^2)(y^2+z^2)\}^{\frac{1}{2}}}-\frac{(1-v_0^2)y^2}{\{x_0^2+(1-v_0^2)(y^2+z^2)\}^{\frac{3}{2}}}\right].$$

Integrating as in the previous cases, we have

$$-f\cdot\frac{6}{5}\cdot\frac{e^2}{2a}\left\{\frac{1}{v_0}\log\frac{1+v_0}{1-v_0}-\frac{(1-v_0^2)\partial}{2v_0\partial v_0}\left(\frac{1}{v_0}\log\frac{1+v_0}{1-v_0}\right)\right\}$$

or

$$-f\cdot\frac{6}{5}\cdot\frac{e^2}{2av_0^2}\left\{\frac{1+v_0^2}{2v_0}\log\frac{1+v_0}{1-v_0}-1\right\},$$

which gives Abraham's expression for the transverse mass.

CORNELL UNIVERSITY,
March, 1908.

THE RECENTLY DISCOVERED MANUSCRIPT OF ARCHIMEDES.

PROFESSOR J. L. Heiberg has published two important accounts of his recent discovery of a new manuscript of Archimedes, both of which are of great interest to mathematicians. The first of these accounts is printed in volume 42 of *Hermes*. It contains the Greek text of a lost treatise of Archimedes, which is recovered nearly complete in the newly found manuscript. A German translation of the Greek text, and an interesting commentary by Zeuthen, is printed by Heiberg in the *Bibliotheca Mathematica*, volume 7, page 321.

Professor Heiberg's critical study of Archimedes has extended over a period of more than thirty years. His dissertation, "Quæstiones Archimedeæ" (Copenhagen, 1879), is constantly referred to by students of Archimedes, both on account of its scholarly critique of Archimedes's work, and on account of the innumerable references to Archimedean literature which are there brought together for the first time. In 1880–1881 Heiberg published the definitive edition of Archimedes, with which his name is usually associated. It was while at work upon a second edition of this book that Heiberg's attention was directed to a palimpsest manuscript of mathematical content recently catalogued in a cloister at Constantinople. Failing in an attempt to have the manuscript sent to him at Copen-

hagen, Heiberg spent the summer of 1906 in Constantinople. Investigation soon showed that the manuscript contained important new material, the study of which would cost more time than was then at Heiberg's disposal. Consequently photography was used wherever possible to copy the pages containing the new material. A facsimile is printed on page 235 of the volume of *Hermes* referred to above.

The contents of the palimpsest are as follows: (1) Large portions of the previously known writings of Archimedes, the Sphere and Cylinder and On Spirals, nearly complete, parts of Measurement of the Circle, and Equilibrium of Planes; (2) a large part of the book On Floating Bodies, which was hitherto extant only in a Latin translation, so that numerous defects and lacunæ in this book can now be healed; (3) a hitherto unknown treatise quoted by Heron and referred to by Suidas, the title of which we will translate "A Method." This book constitutes the most important find of new material. There is also the beginning of "'Αρχιμήδους Στομαχ(ικόν)" which Heiberg says is a sort of "Chinese game," of which there is not enough remaining to patch up the meaning of the original.

The recovery of this lost work of Archimedes must be considered of the very foremost importance. Of course, the recovery of any work of so original a genius as Archimedes must be considered important. The special significance, however, of the present find consists in the fact that here Archimedes admits us to his mathematical workshop, and points out to us, not only how his discoveries were made, but how other discoveries may be made in the future by the method of research which he discloses. It is a common complaint that the Greek mathematics that has come down to us yields no trace of the process of discovery used by the investigator. The motto of the Greek school might well have been: "Results and not processes are for the public." This book of Archimedes is the one known exception, for herein he relates in charming detail the story of the discovery of some of the most fundamental and most interesting theorems of geometry.

Archimedes sends "A Method" to Eratosthenes, in order to make him acquainted with his method of discovery. It is a conscious attempt, not only to hand Eratosthenes the demonstration of certain theorems which Archimedes had propounded to him, without proof; but to give to Eratosthenes the knowledge of a method of research by which many new truths might

be expected to be discovered. At this point it is probably best to let Archimedes speak for himself. It is a matter of great interest that the letters which Archimedes prefixed to his different mathematical treatises have come down to us. These are all of very great value in showing us the personal side of his work and in coordinating the different lines of inquiry that absorbed his attention. The present book is no exception, and the letter to Eratosthenes which opens the book is preserved intact. It reads as follows:

"*ARCHIMEDES TO ERATOSTHENES, GREETING:* I formerly sent you some theorems discovered by myself with the request that you find the proofs which I provisionally withheld. The theorems which I sent to you were as follows:

1. If a cylinder be inscribed in a rectangular prism with square base, so that its bases lie in the opposite squares, and so that elements touch the remaining planes of the prism, and if a plane be passed through the center of the circle which is the base of the cylinder and a side of the square lying in the opposite face of the prism, then this plane cuts off a segment of the cylinder which is bounded by two planes (the cutting plane and the plane wherein lies the base of the cylinder) and by the portion of the cylindrical surface lying between the planes just mentioned; this segment of the cylinder is $\frac{1}{6}$ of the whole prism.
2. If there be inscribed in a cube a cylinder which has its bases in opposite squares and touches the other four faces of the cube, and, moreover in the same cube a second cylinder be inscribed which has its bases in two other squares, and touches the other four faces of the cube, the solid enclosed by the cylindrical surfaces (which solid is common to the two cylinders), is equal to $\frac{2}{3}$ of the entire cube.

"These theorems are essentially different from those communicated by me earlier; the former bodies, the conoids and spheroids and their segments, we compared with the volumes of cones and cylinders, but none of them were found to be equal to a body bounded by planes; on the other hand, each of the present solids, bounded by two planes and cylindrical surfaces, was found equal to solids bounded by planes. I send the proof of these theorems in this book.

"However, since I see that you are an excellent scholar, and that you are not only a prominent student of philosophy, but also a lover of mathematical research, I have deemed it well to explain to you and put down in this same book a special method whereby the possibility will be offered you to investigate any mathematical question by means of mechanics. And I am convinced that it is quite as useful in the proof of the theorems themselves; for many of them that were formerly only clear to me through mechanics were afterwards proved by geometry, although the treatment by the former method was not founded on demonstration: indeed, if one has previously gotten a conception of the problem by this method, it is easier to produce the proof than to find it without a provisional conception. Likewise, if one considers the known theorems whose demonstrations were first found by Eudoxus, namely that the cone and the pyramid are respectively one third of the cylinder and prism having the same base and equal altitudes, one must award no slight credit to Democritus who first obtained expression for these truths without a proof. We are also in the position of having previously discovered in the same way the theorems now to be published, and we feel obliged to make the method known, partly because we have previously spoken of them, lest it be believed that we have spread an idle tale, partly in the conviction that there will be instituted thereby a matter of no slight utility in mathematics. In fact, I believe that living or future investigators will discover other theorems which have not occurred to us by the method here discussed.

"We will first explain a theorem that first became clear to me by means of mechanics — to wit that a segment of a parabola is $\frac{4}{3}$ the triangle that has the same base and equal altitude, but thereafter will be given in turn the several theorems found by the method described; and at the conclusion of the book we present the geometrical proof of the theorems above mentioned."

After this introduction, Archimedes gives a list of eight lemmas which he will use in the following demonstrations. All of these are theorems concerning the position of the center of gravity of known geometrical figures. The first five are found in his treatise "On the equilibrium of planes." No. 6, "The center of gravity of a circle is its center"; No. 7, "The center of gravity of a cylinder is the middle point of its axis"; No. 8, "The center of gravity of a cone divides the axis so that the

portion adjacent to the apex is thrice the portion adjacent to the base" are not found in any of his existing works. They may be contained in a lost work on centers of gravity, referred to by Simplicius, for Archimedes states that these first eight have already been published. He then states another preliminary theorem, which is the same as "On conoids," 1.

The new method of Archimedes is nearly identical with the method of the integral calculus. Many of the processes used by the ancients had the semblance of the method of the infinitesimal calculus, but the present manuscript shows that Archimedes had taken the decisive step in his method of investigation and had done it consciously. In this work he asserts that surfaces are to be considered as made up of lines, and that solids of revolution are "filled up" by circles. It is true that he expressly admits that his method is not demonstration, but he is thoroughly convinced that the results obtained by it are correct. One must emphasize the fact that the conception and use of the infinite was prohibited in Greek mathematics. The paradoxes of Zeno and the sophists had overfrightened the mathematicians of the Greek school and they had found their only safety in their ingenious theory of proportion and the method of exhaustions. It is an exceedingly interesting fact that Archimedes dared to work with the new idea in his preliminary investigations and it is even more remarkable that he was willing to communicate it to his fellow workers. How very strict the mathematical traditions were on this point is well illustrated by the fact that Archimedes not only gives a geometrical proof for the volume of the segment of a cylinder in the form of the infinitesimal method suggested by his mechanical proof, but he afterwards adds a demonstration in the precise form of a regular exhaustion proof, in order that no one could question its validity.*

Archimedes has abundantly shown that he possessed the genius necessary to formulate the infinitesimal method on a sound basis. He evidently made no attempt of this sort, and it does not appear certain whether his lack of effort in this direction was due to the fact that he saw no means of avoiding in all cases the use of the principles of mechanics in applying the infinitesimal method, and hence the impossibility of a

* These comments are substantially those of Heiberg. For many references given by Heiberg see *Hermes*, l. c., p. 302. The last sentence of above paragraph refers to theorems XI, XII, XIII of Archimedes's text.

demonstration purely geometrical; or whether he was deterred from such effort on account of adhesion to the traditions of the Greek school. Besides these considerations, there is the strong native desire of great constructive intellects to leave their investigations in the synthetic form — so strongly apparent in the case of Newton and by no means extinct in later days. My own conviction, after some years of familiarity with Archimedes's writings, is that Archimedes was not willing to rest a geometrical demonstration upon mechanical principles, except provisionally, and that his reason was not that rigor could not be secured in that direction, but that there was a scientific propriety, as well as an æsthetic obligation, to keep geometry sufficient unto itself. We all see how such a doctrine fits into the platonic philosophy of Archimedes's time and is in accord with the spirit of the Alexandrian school.

Few realize, except those who have put some study upon Archimedes's work, the tremendously exacting labor that Archimedes put upon himself in placing the demonstrations of his discoveries in the form of exhaustion proofs. To him geometry was the noblest of the sciences. Plutarch may not be exaggerating when he says, "he placed his whole delight in these intellectual speculations, which, without any relation to the necessities of life, have an intrinsic excellence arising from truth and demonstration only."

The first two theorems in the new manuscript of Archimedes suffice to illustrate his method. I have abbreviated or omitted descriptive portions of the demonstrations, but have put in his exact language as far as practicable those portions which are of the greatest importance in conveying his exact point of view.

"Theorem I. Let $\alpha\beta\gamma$ be a segment of a parabola bounded by the straight line $\alpha\gamma$ and the parabola $\alpha\beta\gamma$. Let $\alpha\gamma$ be bisected at δ and let $\delta\beta\epsilon$ be drawn parallel to the diameter of the parabola. Draw $\alpha\beta$ and $\beta\gamma$. Then the segment $\alpha\beta\gamma$ is $\frac{4}{3}$ the triangle $\alpha\beta\gamma$."

"Draw from the points α, γ the line $\alpha\zeta$ parallel to $\delta\beta\epsilon$ and the tangent $\gamma\zeta$. Produce $\gamma\beta$ to κ and make $\kappa\theta = \gamma\kappa$. We then suppose that $\gamma\theta$ is a 'neck yoke'* with middle point κ

*The Greek $\zeta\upsilon\gamma\acute{o}\nu$ literally means yoke. It is the same word as the Latin *jugum* or English *yoke*. It came to be used for beam of a balance and Archimedes uses it as the equivalent of *lever*, by which it is rendered in translations of Archimedes.

and that $\mu\xi$ is any straight line parallel to $\epsilon\delta$. Since $\gamma\beta\alpha$ is a parabola, $\gamma\zeta$ a tangent and $\gamma\delta$ an ordinate, then $\epsilon\beta = \beta\delta$; in fact this is proved in the Elements. For this reason, and since $\zeta\alpha$ and $\mu\xi$ are parallel to $\epsilon\delta$, we know that $\mu\nu = \nu\xi$, $\zeta\kappa = \kappa\alpha$. Since $\gamma\alpha : \alpha\xi = \mu\xi : \xi o$ (in fact this was proved in a lemma [Quad. Parab. 5]), we have $\gamma\alpha : \alpha\xi = \gamma\kappa : \kappa\nu$, and since $\gamma\kappa = \kappa\theta$, therefore $\theta\kappa : \kappa\nu = \mu\xi : \xi o$. Now ν is the center of gravity of the

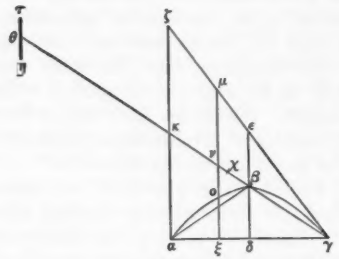

FIG. 1.

straight line $\mu\xi$, since $\mu\nu = \nu\xi$; hence if we construct $\tau\eta = \xi o$ with θ as its center of gravity, so that $\tau\theta = \theta\eta$, then the straight line $\tau\theta\eta$ must be in equilibrium about κ with $\mu\xi$ in the position where it is; for $\theta\nu$ is divided in the inverse ratio of the weights $\tau\eta$ and $\mu\xi$ and $\theta\kappa : \kappa\nu = \mu\xi : \eta\tau$; therefore κ is the center of gravity of the combined weights. In the same way all lines which are drawn in the triangle $\zeta\alpha\gamma$ parallel to $\epsilon\delta$ are in equilibrium, in the position where they are, with their segments intercepted by the parabola, provided these latter be displaced to θ, so that κ will be the center of gravity of the combined weights. Since the lines of the triangle $\gamma\zeta\alpha$ make up the triangle, and since the lines corresponding to ξo make up the segment of the parabola $\alpha\beta\gamma$, therefore the triangle $\zeta\alpha\gamma$, in the place where it is, is in equilibrium about the point κ, with the segment of the parabola, if this latter be displaced so as to have θ as center of gravity; hence κ is the center of gravity of the combined weights.

"Now let $\gamma\kappa$ be so divided at χ that $\gamma\kappa = 3\kappa\chi$; then χ is the center of gravity of the triangle $\alpha\zeta\gamma$; for this is proved in the theory of equilibrium (De plan. æquil. I, 14). Now the triangle $\zeta\alpha\gamma$, in the place where it is, is in equilibrium about κ

with the segment $\beta\alpha\gamma$, if this be displaced so that θ is its center of gravity, and the center of gravity of the triangle $\zeta\alpha\gamma$ is χ; therefore the triangle $\alpha\zeta\gamma$: segment $\alpha\beta\gamma$ displaced to θ as center of gravity :: $\theta\kappa : \kappa\chi$. But $\theta\kappa = 3\kappa\chi$; therefore the triangle $\alpha\zeta\gamma = 3$ segments $\alpha\beta\gamma$. But the triangle $\zeta\alpha\gamma = 4$ triangles $\alpha\beta\gamma$, since $\zeta\kappa = \kappa\alpha$ and $\alpha\delta = \delta\gamma$; therefore the segment $\alpha\beta\gamma = 4/3$ the triangle $\alpha\beta\gamma$."

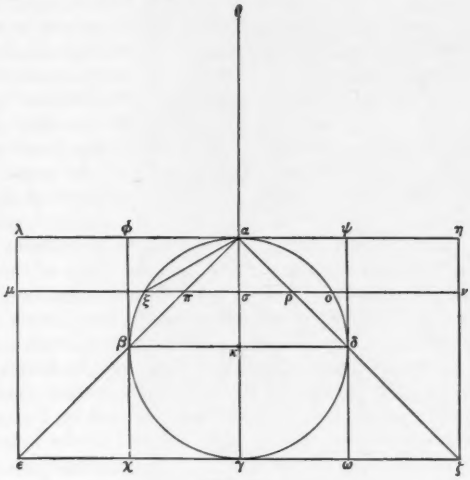

FIG. 2.

"Theorem II. That the sphere is four times as great as a cone whose base is equal to a great circle of the sphere, but whose altitude equals the radius of the sphere; and that a cylinder whose base is equal to the great circle of the sphere, but whose altitude is equal to the diameter of the circle, is one and one half times as great as the sphere, can be explained by the present method in the following manner:

"Let there be (Fig. 2) a sphere and two mutually perpendicular diameters $\alpha\gamma$, $\beta\delta$ of the great circle $\alpha\beta\gamma\delta$; there is a circle of the sphere perpendicular to $\alpha\beta\gamma\delta$ whose diameter is $\beta\delta$. On this circle let a cone be constructed whose apex is α and afterwards let its convex surface be produced until the conical surface is cut by the plane passing through γ parallel to the base

of the first cone, in consequence a circle is intercepted which is perpendicular to $\alpha\gamma$ and whose diameter is $\epsilon\zeta$. Erect a cylinder on this circle whose axis is $\alpha\gamma$ and having elements $\epsilon\lambda$ and $\zeta\eta$. Produce $\gamma\alpha$ and make $\alpha\theta = \gamma\alpha$, and suppose $\gamma\theta$ to be a lever whose fulcrum is α; moreover let any line, as $\mu\nu$, be drawn parallel to $\beta\delta$; it cuts the circle $\alpha\beta\gamma\delta$ in ξ and o, the diameter $\alpha\gamma$ in σ, the line $\alpha\epsilon$ in π and $\alpha\zeta$ in ρ. Let a plane be passed through the line $\mu\nu$ perpendicular to $\alpha\gamma$; it will cut a circle from the cylinder of diameter $\mu\nu$; it will cut the sphere in a circle of diameter ξo; it will cut the cone $\alpha\epsilon\zeta$ in a circle of diameter $\pi\rho$. Now since $\gamma\alpha \times \alpha\sigma = \mu\sigma \times \sigma\pi$ (for $\alpha\gamma = \sigma\mu$, $\alpha\sigma = \pi\sigma$) and $\gamma\alpha \times \alpha\sigma = \alpha\xi^2 = \xi\sigma^2 + \sigma\pi^2$, therefore $\mu\sigma \times \sigma\pi = \xi\sigma^2 + \sigma\pi^2$; moreover, since $\gamma\alpha : \alpha\sigma = \mu\sigma : \sigma\pi$ and $\gamma\alpha = \alpha\theta$, therefore $\theta\alpha : \alpha\sigma = \mu\sigma : \sigma\pi = \mu\sigma^2 : \mu\sigma \times \sigma\pi$. It has been shown, however, that $\xi\sigma^2 + \sigma\pi^2 = \mu\sigma \times \sigma\pi$; therefore $\alpha\theta : \alpha\sigma = \mu\sigma^2 : \xi\sigma^2 + \sigma\pi^2$. But $\mu\sigma^2 : \xi\sigma^2 + \sigma\pi^2 = \mu\nu^2 : \xi o^2 + \pi\rho^2$ = the circle of cylinder of diameter $\mu\nu$: the circle of cone of diameter $\pi\rho$ + the circle of sphere of diameter ξo. Therefore $\theta\alpha : \alpha\sigma =$ the circle of cylinder : circle of cone + circle of sphere. Therefore, the circle of the cylinder, in the place where it is, is in equilibrium about the point α with the combined circles of diameters ξo, $\pi\rho$, if they be displaced to θ so that θ is the center of gravity of both. In the same way it can be shown that if any other line be drawn in the parallelogram $\zeta\lambda$, parallel to $\epsilon\zeta$, and a plane passed through it perpendicular to $\alpha\gamma$, then the circle intercepted by the cylinder is in equilibrium about α with the combined circles intercepted by the sphere and cone, provided they be displaced and so placed on the lever arm at θ that θ is the center of gravity of both. If, therefore, the cylinder, sphere and cone be filled up with the above mentioned circles, then the cylinder, in the place where it is, will be in equilibrium about α with the combined cone and sphere, if they be displaced and so placed on the lever arm at θ that θ is the center of gravity of both. Now since the solids named are in equilibrium, the cylinder with center of gravity κ, the cone and sphere displaced as stated with center of gravity θ, then $\theta\alpha : \alpha\kappa =$ cylinder : sphere + cone. But $\theta\alpha = 2\alpha\kappa$ and the cylinder is equivalent to three cones (Euclid, Elements XII, 10); therefore 3 cones = 2 cones + 2 spheres. If two cones be taken away from both members, the cone whose axial triangle is $\alpha\epsilon\zeta$ is equal to two spheres. But the cone whose axial triangle is $\alpha\epsilon\zeta = 8$ cones whose axial triangle is $\alpha\beta\delta$, because $\epsilon\zeta = 2\beta\delta$, therefore, the 8 cones named

are equal to two spheres. Consequently the sphere whose great circle is $\alpha\beta\gamma\delta$ is four times as great as the cone whose apex is α and whose base is the circle of diameter $\beta\delta$ perpendicular to $\alpha\gamma$.

"Through β and δ draw the lines $\phi\beta\chi$ and $\psi\delta\omega$ in the prallelogram $\lambda\zeta$ parallel to $\alpha\gamma$ and introduce the cylinder whose bases are circles having the diameters $\phi\psi$ and $\chi\omega$, and whose axis is $\alpha\gamma$. Then the cylinder whose axial parallelogram is $\phi\omega$ is twice as large as the cylinder whose axial parallelogram is $\phi\delta$, and this last is threefold as large as the cone whose axial triangle is $\alpha\beta\delta$, as is proved in the Elements. Therefore the cylinder whose axial parallelogram is $\phi\omega$ is sixfold as large as the cone whose axial triangle is $\alpha\beta\delta$. It has been proved, however, that the sphere of great circle $\alpha\beta\gamma\delta$ is fourfold as large as the same cone; consequently the cylinder is $\frac{3}{2}$ the sphere, which was to be proved.

"Because of this theorem — that a sphere is fourfold as large as the cone whose base is a great circle and whose altitude is the radius of the sphere — the thought occurred to me that the surface of the sphere is fourfold as great as its great circle, because I proceeded from the conception that as a circle is equivalent to a triangle whose base is equal to the circumference of the circle and whose altitude is equal to the radius of the circle, in the same way a sphere is equal to a cone whose base is equal to the surface of the sphere, and whose altitude is equal to the radius of the sphere."

Following the above theorems, Archimedes gives a number of others, all of which are investigated by the same method. Among these are: volume of a spheroid; volume of a paraboloid; center of gravity of a paraboloid; center of gravity of a hemisphere; volume of spherical segment; center of gravity of a spherical segment. Then follows the theorem concerning the volume of the cylindrical segment announced in the introduction. This contains a number of lacunæ. No trace is left of the theorem concerning the solid common to two intersecting cylinders, and of the geometrical proof of the area of parabolic segment, etc., which were promised in the introduction.

The two theorems reproduced above illustrate the method used by Archimedes in the remaining theorems. The method may be summarized as follows: The elements of a solid or surface to be measured are connected by a proportion with the elements of bodies previously considered, and in such manner that

two terms of the proportion shall be straight lines. The straight lines may then be used as "lever arms" to balance the elements or certain combination of elements selected from the new and old bodies. Then if the magnitude and position of the centers of gravity of the old bodies be known, the new body, when displaced so that its center of gravity lies at one end of the lever arm, will be in equilibrium with a known magnitude "in the place where it is" and hence an equality is established. This means, in modern language, that a definite integral is evaluated by means of the mean value of a function defined by means of the center of gravity. On the other hand, if the magnitudes of the bodies under investigation be known, the center of gravity of one of them can be found by the same process. The method requires the introduction of the principle of moments as a postulate of mathematics and a formulation of the doctrine of infinitesimals. Archimedes avoided this by the construction of exhaustion proofs.

A number of matters of considerable historical interest are settled by the present work. It is now known that Democritus and not Eudoxus is to be given credit for the discovery of the relation between the volume of the pyramid and prism and of the cone and cylinder. Evidently Eudoxus was the first to put the demonstrations in the form of an exhaustion proof, but his work is to be compared with Archimedes's geometrical demonstrations constructed after the facts had been developed by means of mechanics. It is also of interest to note that Archimedes states that he discovered the theorem concerning the volume of the sphere previous to the discovery of the theorem expressing the area. His description of the manner in which the idea occurred to him that the surface of the sphere is fourfold its great circle, is of the very greatest interest. One would naturally but erroneously infer from the way the matter is put in his treatise on the Sphere and Cylinder, that the area of the sphere became known to him before the discovery of its volume.

Archimedes makes it clear that he is using this method to "discover" the theorems, and that it is "not founded on demonstration." But in speaking of spheres as "filled up" by circles and of surfaces as "made up" of lines, he is not misusing the method of infinitesimals nor treading on dangerous ground. In fact the elements which "fill up" the magnitudes are always so taken by Archimedes that the process can be immediately satisfied by an exhaustion proof. From this point of view his

scheme may be regarded as analogous to the modern method of infinitesimals when founded upon the doctrine of limits. There is a normal and systematic procedure, although tedious and laborious, for converting the mechanical proofs into exhaustion proofs. Therefore Archimedes may be regarded as having taken the decisive step in founding a method which in essential respects is that of the integral calculus. If he had been like many modern mathematicians, he would have omitted the exhaustion proofs altogether, but would have added to each of his mechanical proofs a set phrase like this: "It is easy to see that an exhaustion proof may be constructed in the usual manner. This is left as an exercise for the reader."

CHARLES S. SLICHTER.

SHORTER NOTICES.

Mehrdimensionale Geometrie, II Teil, *Die Polytope*. Von PROFESSOR DR. P. H. SCHOUTE. Leipzig, G. J. Göschen (Sammlung Schubert XXXVI). 1905. ix + 326 pp.

THE second (and final) volume of this work, like the first, is worthy to be associated with the other excellent books of the Schubert collection. Comparatively little of the subject matter is new, but a large number of interesting and useful results have been gathered together in a convenient form. The entire volume is devoted to the treatment of the polytop, which the author defines, for space of n dimensions, as any portion of that space enclosed in any manner whatever. The first 262 pages treat the linear polytop, *i. e.*, one bounded by flat spreads of $n-1$ dimensions (R_{n-1}'s); while the remaining 64 pages are concerned with the hypersphere, cone, cylinder, and rotation spread.

Under the heading "Topologische Einleitung," the first section treats (among other things) the simplex, and its various sections and projections; the question of a general classification of polytops; the definition of hyper-pyramids and prisms and the n-dimensional analogues of other special polyhedra, such as the truncated pyramid and prism, the frustums, etc.; and finally discusses the Euler law and its n-dimensional extension. In this treatment of the Euler law, thirteen pages are devoted to the well-known three-dimensional case, four different methods

of proof being given, while the extension to n dimensions is disposed of in five pages. In view of the difficulty which the author mentions in the preface of reducing the volume to the required number of pages, it would seem that some space might have been saved to advantage at this point.

The second section is devoted to "Massverhältnisse." Congruence, both positive and negative, and similarity are defined, first for the simplex and then for the general linear polytop. Attention is called to three points metrically related to the simplex, viz., the in-center, the circumcenter, and the centroid or center of gravity. One regrets here that the author was forced to omit his originally intended paragraph on the geometry of the simplex. The orthocenter, so closely related to the three points mentioned, might well have been included. The important subject of the section is that of the content (Inhalt) of polytops. Formulas are deduced for the content of various types of polytops in the following order: The rectangular parallelotop, the general parallelotop, the prism, the simple pyramid, the simplex, etc., the general procedure being similar to the treatment in our elementary geometries. A convenient expression in determinant form for the content of a simplex in terms of the lengths of its $\binom{n+1}{2}$ edges is deduced.

"Reguläre Polytope" is the subject of the third section; and here again, in a work on higher-dimensional geometry, much less attention might have been given to the two- and three-dimensional cases. The larger part of the fifteen and forty-five pages devoted respectively to the regular polygon and regulae polyhedron might have been more advantageously used for some of the material which the author was forced to omit in condensing his manuscript. The treatment of the higher-dimensional polytops furnishes, in the estimation of the writer, the two most interesting chapters of the book. In one chapter the six four-dimensional figures are considered, the simplest sets of rectangular cartesian coordinates of the vertices are given for each, the Schlegel diagrams and various three-dimensional sections are studied, and the group connected with each figure is given. In the next chapter the three regular polytops which exist in space of any number of dimensions are studied. In a convenient little table one finds, for each of the three figures, formulas for the radii of the n associated hyper-spheres, the angle between two adjacent R_{n-1}'s, the content, and the boundary content (Oberfläche), all in terms of the length of an edge. The

general expression for the order of the group associated with each of these figures is also given. The omission of a phrase from the theorem at the top of page 256 is easily corrected by reference to the formula at the bottom of the preceding page.

The fourth and final section treats "Die runden Polytope." In the work on the hyper-sphere one notices such topics as, the sphere passing through $n + 1$ points, the sphere touching $n + 1$ R_{n-1}'s, the sphere touching $n + 1$ other spheres, the configuration of the centers of similitude of $n + 1$ spheres, the content and surface content of the sphere, and the content of the spherical sector and segment. The hyper-cone and cylinder are similarly treated. Under general rotation figures, one finds the quadric spreads generated by revolving a flat spread about a flat spread as an axis, the torus spreads and the Guldin spreads obtained by revolving hyperspheres and linear polytops respectively.

Since the book is largely a compilation of previously known results, one regrets that references to the literature of the subject are not more numerous and specific. This second volume is better than the first in this respect, but still leaves much to be desired.

The book will doubtless prove to be a valuable reference work to those who are interested in, and have use for, the metrical formulas of higher-dimensional geometry; but many readers will doubtless share with the present writer a regret that the author's point of view has been so largely metrical, both in his choice of topics and in his method of treatment.

W. B. CARVER.

Theory of the Algebraic Functions of a Complex Variable. By J. C. FIELDS. Berlin, Mayer & Müller, 1906. v + 186 pp.

THE work before us is not intended as a treatise or text-book on the theory of algebraic functions along any of the well-established lines of treatment. It is, on the contrary, a new and distinctive mode of approach to this class of functions, although grounded on principles which in their essence are already familiar. The methods employed are purely algebraic, we might almost say arithmetic, in character, and in this respect the influence of Weierstrass may be said to predominate.

The fundamental idea on which the work is based is the notion of "order of coincidence." A given class of algebraic functions is defined as usual by a rational expression in (z, v)

the dependence of v on z being determined by an algebraic equation $F(v, z) = 0$ of degree n in v. The n branches of the function are assumed in the form $v = P_k(z - a)$ $(k = 1, 2, \cdots, n)$ in the vicinity of a given point $z = a$, the exponents in the power series being rational numbers only a finite number of which are negative. The order of coincidence of any function $H(z, v)$ with the branch $v - P_k = 0$ is then defined to be the least exponent obtained in developing H in powers of $z - a$, after substituting for v the series $P_k(z - a)$. For each point a there is accordingly a set of n (or fewer) numbers constituting the orders of coincidence of the given function with the several branches of the curve $F(v, z) = 0$. This set of numbers will be zero except for a finite number of points a. These sets of non-vanishing orders of coincidence are studied and the conditions which they must satisfy are determined.

On the other hand, if the orders of coincidence be completely given in advance, the function H is thereby conditioned, and it is then required to determine its form and properties. Chapter VI, for example, is devoted to finding the relation between the degree of a function and its orders of coincidence at infinity.

The methods of the author, growing as they do out of a single fundamental idea, are naturally characterized by a high degree of simplicity, unity, and generality. This is especially observable in Chapters XIII, XIV, and XV where the efficiency of the new treatment and its culminating "complementary theorem" of Chapter XII is tested by deducing for any algebraic relation in (z, v), reducible as well as irreducible, the Riemann-Roch and related theorems and formulas, and the general forms and properties of the ϕ-functions and abelian integrals. The brief discussion of the abelian integrals contained in the last chapter is merely intended as a suggestion of ways in which the previous theory might be utilized in the study of this class of transcendental functions.

As remarked in the preface, the methods of this work, at least in part and with suitable modifications, are applicable outside the field of algebraic functions and to functions of any number of variables.

The book is printed in a style uniform with the elegant edition of Weierstrass's works issued by the same publishers.

J. I. HUTCHINSON.

NOTES.

THE April number (volume 9, number 2) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Representations of the general symmetric group as linear groups in finite and infinite fields," by L. E. DICKSON; "Surfaces with isothermal representation of their lines of curvature and their transformations," by L. P. EISENHART; "The equilong transformations of space," by J. L. COOLIDGE; "Concerning linear substitutions of finite period with rational coefficients," by A. RANUM; "On hypercomplex number systems belonging to an arbitrary domain of rationality," by R. B. ALLEN; "On the asymptotic character of the solution of certain linear differential equations containing a parameter," by G. D. BIRKHOFF; "On the holomorph of the cyclic group of order p^m," by G. A. MILLER; "On non-measurable sets of points," by E. B. VAV VLECK.

THE April number (volume 30, number 2) of the *American Journal of Mathematics* contains: "Concerning systems of conics lying on cubic, quartic, and quintic surfaces," by C. H. SISAM; "On the canonical forms and automorphs of ternary cubic forms," by L. E. DICKSON; "The elliptic cylinder function of class K," by W. H. BUTTS; "On elliptic modular equations for transformations of orders 29, 31, 37," by A. BERRY; "On translation surfaces connected with a unicursal quartic," by J. EIESLAND.

AT the meeting of the London mathematical society held on March 12 the following papers were read: By E. B. ELLIOTT, "On the projective geometry of some covariants of a binary quintic"; by W. H. YOUNG, "On the inequalities connecting the double and repeated upper and lower integrals of a function of two variables"; by W. F. SHEPPARD, "On the operational expression of Taylor's theorem"; by H. A. P. DE S. PITTARD, "A proof of a theorem of Fermat's"; by M. J. M. HILL, "On a formula for the sum of a finite number of terms of the hypergeometric series, whose fourth element is unity."

THE summer meeting of the American association for the advancement of science will be held at Dartmouth College, Hanover, N. H., during the week beginning June 29.

THE tenth regular meeting of the association of teachers of mathematics in the Middle States and Maryland was held at the Woman's College, Baltimore, March 14. The following papers were read: By L. S. HULBERT, "Undergraduate instruction in mathematics"; by I. J. SCHWATT, "Our duty as teachers"; by W. H. JACKSON, "Notes on the teaching of mathematics in English preparatory schools and colleges"; by F. MORLEY, "A test of elementary text books in geometry"; by A. J. GMINDER, "The history of mathematical symbolism."

THE Manchester mathematical society was organized at Manchester, England, February 19, 1908, with an initial membership of 60. Professor H. LAMB was elected president, and Mr. H. BATEMAN secretary. Meetings will be held bimonthly. For the present the society will not undertake the publication of its proceedings.

THE following courses in mathematics are announced by German universities for the summer semester of 1908.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: On space curves and curved surfaces, four hours; Calculus of variations, four hours; Applications of elliptic functions, two hours; Seminar, two hours; Colloquium, two hours. — By Professor G. FROBENIUS: Theory of determinants, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Elementary analysis, four hours; Theory of abelian functions, four hours; Seminar, two hours. — By Professor G. HETTNER: Introduction to the theory of ordinary differential equations, two hours. — By Professor J. KNOBLAUCH: Applications of elliptic functions, four hours; Calculus, one hour. — By Professor R. LEHMANN-FILHÈS: Analytic mechanics, four hours. — By Professor E. LANDAU: Differential calculus, four hours; Distribution of prime numbers, four hours. — By Dr. I. SCHUR: Theory of algebraic equations, II, four hours; Analytic geometry, four hours.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Encyclopedia of geometry, four hours; Seminar, two hours. — By Professor D. HILBERT: Principles of mathematics, four hours; Seminar, two hours. — By Professor H. MINKOWSKI: Analytic geometry, four hours; Fourier series and definite integrals, two hours; Seminar, two hours. — By Professor C.

RUNGE: Differential and integral calculus with exercises, six hours; Graphical statics, one hour; Seminar, two hours. — By Professor L. PRANDTL: Theory of elasticity and rigidity, three hours; Introduction to the theory of machines, one hour; Seminar, two hours. — By Professor G. HERGLOTZ: Equilibrium and motion of liquids under gravity, three hours; Selected chapters of celestial mechanics, three hours. — By Professor E. ZERMELO: Mathematical logic, two hours. — By Dr. C. CARATHÉODORY: Functions of real variables, two hours; Historical introduction to the calculus of variations, one hour. — By Dr. P. KOEBE: Elementary differential equations, six hours. — By Dr. O. TOEPLITZ: Introduction to the theory of integral equations, three hours; Proseminar, two hours. — By Professor F. BERNSTEIN: Probabilities, two hours; History of mathematics, two hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Analytic mechanics, four hours. — By Professor O. HÖLDER: Applications of elliptic functions, three hours; Elliptic modular functions, three hours; Seminar, two hours. — By Professor F. HAUSDORFF: Differential equations, four hours. — By Professor K. ROHNS: Algebraic curves, four hours; Descriptive geometry, I, two hours. — By Professor H. LIEBMANN: Plane analytic geometry, four hours; Vector analysis with applications, two hours.

THE following advanced courses are announced by the various American universities for the academic year 1908–1909:

CORNELL UNIVERSITY. — By Professor J. MCMAHON: Theory of sound, two hours; Electricity, two hours. — By Professor J. I. HUTCHINSON: Theory of functions of a complex variable, three hours. — By Professor V. SNYDER: Higher geometry, three hours. — By Professor W. B. FITE: Theory of groups, three hours. — By Dr. F. R. SHARPE: Theory of potential and Fourier's series, three hours; Elliptic functions, two hours, first half year. — By Dr. W. B. CARVER: Projective geometry, three hours. — Dy Dr. A. RANUM: Elementary differential equations, two hours; Higher algebra, two hours. — By Dr. D. C. GILLESPIE: Advanced calculus, three hours; Integral equations, two hours, second half year. — By Dr. C. F. CRAIG: Advanced analytic geometry, three hours;

Partial differential equations, two hours, first half year. — By Dr. F. W. OWENS: Solid analytic geometry, two hours.

The Oliver mathematical club will meet weekly.

INDIANA UNIVERSITY. — By Professor R. J. ALEY: Advanced calculus, three hours (*a*, *w*, *s*); Higher algebra, two hours (*a*, *w*); Algebra of quantics, three hours (*s*). — By Professor S. C. DAVISSON: Ordinary differential equations, three hours (*a*, *w*); Functions of a complex variable, three hours (*s*); Fourier's series and integrals, three hours (*a*); Modern analytic geometry, two hours (*w*, *s*). — By Professor D. A. ROTHROCK: Quaternions, three hours (*a*); Partial differential equations, three hours (*w*, *s*). — By Professor U. S. HANNA: Elliptic integrals and functions, two hours (*a*, *w*); Infinite series and products, three hours (*s*). — By Dr. C. HASEMAN: Mathematical theory of elasticity, three hours (*a*, *w*); Theory of potential, three hours (*s*). [*a*, *w*, *s* above indicate autumn, winter, and spring terms.]

YALE UNIVERSITY. — By Professor J. PIERPONT: Introduction to the theory of functions, two hours; Projective geometry, two hours; Advanced mechanics, two hours; Advanced theory of functions, two hours. — By Professor P. F. SMITH: Advanced analytic geometry, two hours; Continuous groups of transformations, two hours. — By Professor E. W. BROWN: Mechanics, two hours; Advanced calculus, three hours; Celestial mechanics, two hours. — By Professor H. E. HAWKES: Algebra and analytic geometry, two hours; Theory of equations, two hours. — By Professor M. MASON: Linear differential equations, two hours; Calculus of variations, one hour. — By Dr. L. I. HEWES: Differential equations, one hour; Graphical and numerical computation, one hour. — By Dr. W. A. GRANVILLE: Differential geometry, two hours.

THE following advanced summer courses are announced:

UNIVERSITY OF CHICAGO (summer quarter, June 13 to August 28, 1908).— By Professor H. S. WHITE: Theory of elliptic functions, four hours; Higher plane curves, four hours. — By Professor H. E. SLAUGHT: Theory of definite integrals, four hours; Analytic geometry, five hours. — By Professor J. W. A. YOUNG: Differential calculus, five hours; Pedagogy of secondary mathematics, four hours. — By Professor L. E.

DICKSON: Theory of equations, four hours. — By Professor G. A. BLISS: Differential geometry from the Lie standpoint, four hours; Integral calculus, five hours. — By Dr. A. C. LUNN: Analytic mechanics, four hours; Higher algebra, five hours.

COLUMBIA UNIVERSITY (summer session, July 7 to August 14). — By Professor JAMES MACLAY: Advanced calculus, five hours. — By Professor W. B. SMITH: Theory of functions of a complex variable, five hours. — By Professor EDWARD KASNER: Differential equations, five hours. — By Dr. G. H. LING: Modern higher algebra, five hours.

INDIANA UNIVERSITY (summer session, June 25 to Sept. 4). — By Professor R. J. ALEY: History of mathematics, three hours; Ordinary differential equations, four hours. — By Professor S. C. DAVISSON: Advanced integral calculus, five hours; Theory of surfaces, five hours. — By Professor D. A. ROTHROCK: Solid analytic geometry, three hours.

UNIVERSITY OF PENNSYLVANIA (summer session, July 6 to August 15). — By Professor I. J. SCHWATT: Definite integrals, five hours. — By Professor G. H. HALLETT: Theory of functions of a complex variable, five hours. — By Professor F. H. SAFFORD: Differential equations, five hours.

THE following Smith's prizemen are announced for the year 1908: W. J. HARRISON (Clare College) for his essay, "Problems in the wave motion of viscous liquids"; J. E. LITTLEWOOD (Trinity College) for his essay, "On the asymptotic behavior of integral functions of zero order, and allied functions"; J. MERCER (Trinity College) for his essay, "On the solution of ordinary linear differential equations having doubly periodic coefficients." Five other essays were declared worthy of honorable mention.

THE Hon. BERTRAND RUSSELL has been elected a fellow of the Royal society of London.

PROFESSOR P. STÄCKEL, of the technical school at Hanover, has accepted a call as professor of mathematics at the technical school at Karlsruhe.

PROFESSOR V. VOLTERRA, of the University of Rome, has been elected foreign member of the royal academy of sciences of Stockholm.

PROFESSOR H. POINCARÉ, of the University of Paris, has been elected to membership in the French academy, as successor to the late M. SULLY PRUDHOMME. Professor Poincaré has been a member of the academy of sciences since 1887.

THE Institute of France has awarded the Binoux prize (for the history of the sciences) for the current academic year to Professor GINO LORIA, of the University of Genoa.

DR. E. HILB has been appointed docent in mathematics at the University of Erlangen.

AT the University of Nebraska, Professor C. C. ENGBERG has been promoted to a full professorship of applied mathematics, and Professor W. C. Brenke has been promoted to an assistant professorship of mathematics.

MR. B. F. MORRELL has been appointed professor of mathematics at Fort Worth University, Fort Worth, Texas.

AT Harvard University, Dr. J. L. Coolidge has been promoted to an assistant professorship of mathematics.

MR. P. B. TURNER has been appointed tutor in mathematics at the College of the City of New York.

PROFESSOR C. H. ASHTON, of the University of Kansas, has been granted leave of absence for the next academic year, and will study in Munich.

PROFESSOR L. LINDELÖF, of the University of Helsingfors, died March 3, at the age of 70 years.

DR. M. P. H. LAURENT, examiner in mathematics at the Ecole polytechnique of Paris, died March 2, at the age of 67 years.

PROFESSOR L. WEDEKIND, of the technical school at Karlsruhe, died February 8, at the age of 69 years.

PROFESSOR A. E. v. BRAUNMÜHL, of the technical school at Munich, died March 11, at the age of 55 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BJÖRNBO (A. A.). See VERNERUS (I.).

CAMACHO Y SANJURJO (J. J.). Invariantes; elementos de geometría analítica. Sevilla, Diaz, 1907. 23 pp.

DINGLER (H.). Beiträge zur Kenntnis der infinitesimalen Deformationen einer Fläche. (Diss.) München, 1907. 8vo. 62 pp.

DRESSLER (H.). Die Lehre von der Funktion. Theorie und Aufgabensammlung für alle höheren Lehranstalten. (Mittelschulen.) Leipzig, Dürr, 1908. 8vo. 93 pp. M. 1.60

FULCO (P.). I coefficienti delle equazioni differenziali lineari omogenee, ammettenti fra i loro integrali particolari funzioni $\alpha(z)$. Memoria. Lodi, Wilmant, 1908. 8vo. 10 pp.

GEIGENMÜLLER (R.). Leitfaden und Aufgabensammlung zur höheren Mathematik. Für technische Lehranstalten und den Selbstunterricht bearbeitet. Vol. II. Die höhere Analysis oder Differential- und Integralrechnung. 6te Auflage. Mittweida, 1908. 8vo. 8 + 339 pp. Cloth. M. 7.00

HEGER (R.). Analytische Geometrie auf der Kugel. (Sammlung Schubert, LIV.) Leipzig, Göschen, 1908. 12mo. 7 + 152 pp. Cloth. M. 4.40

HASEMAN (C.). Anwendung der Theorie der Integralgleichungen auf einige Randwertaufgaben in der Funktionentheorie. (Diss.) Göttingen, 1907. 8vo. 46 pp.

JUNKER (F.). Repetitorium und Aufgabensammlung zur Integralrechnung. Leipzig, 1907. 12mo.

KEYSER (C. J.). Mathematics. (A lecture delivered at Columbia University.) New York, Columbia University Press, 1907. 8vo. 44 pp. $0.25

KÖSTLIN (E.). Ueber eine Deutung der Gleichung, die zwischen dem Bogen und dem Neigungswinkel der Tangente im Endpunkt des Bogens einer ebenen Kurve besteht. (Diss.) Tübingen, 1907. 8vo. 60 pp.

LÖFFLER (E.). Beiträge zur Theorie der Schnittpunkte algebraischer Kurven. (Diss.) Tübingen, 1907. 8vo. 49 pp.

RATSCHLÄGE und Erläuterungen für die Studierenden der Mathematik und Physik an der Universität Göttingen. Herausgegeben von der Direktion des mathematisch-physikalischen Seminars. Neue Auflage. Mit einem Anhang: Statuten des mathematischen Lesezimmers. Leipzig, Teubner, 1907. 8vo. 31 pp.

REINECK (A.). Die Verwandtschaft zwischen Kugelfunktionen und Besselschen Funktionen. (Diss., Bern.) Halle a. S., 1907. 8vo. 72 pp.

RONCO (N.). Studio geometrico delle reciprocità; tesi per la laurea in matematica pura. Torino, Streglio, 1907. 8vo. 45 pp.

SCHOENFLIES (A.). Die Entwickelung der Lehre von den Punktmannigfaltigkeiten. Bericht, erstattet der deutschen Mathematiker-Vereinigung. 2er Teil. (Jahresbericht der deutschen Mathematiker-Vereinigung, Ergänzungsband II.) Leipzig, Teubner, 1908. 8vo. 10+331 pp.

SCHÜBEL (H.). Aufstellung von nicht-euklidischen Minimalflächen. (Diss.) München, 1906. 8vo. 47 pp.

VERNERUS (J.). Ioannis Verneri de triangulis sphæricis libri quatuor, de meteoroscopiis libri sex, cum prooemio G. I. Rhetici. I. De triangulis sphæricis. Herausgegeben von A. A. Björnbo. (Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, begründet von M. Cantor, 24es Heft, 1te Abteilung.) Leipzig, Teubner, 1907. 8vo. 12+184 pp.

II. ELEMENTARY MATHEMATICS.

BARDEY (E.). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik. 2er Teil. (Für die Oberklassen neunstufiger Anstalten.) Bearbeitet von H. Hartenstein. Leipzig, Teubner, 1907. 8vo. 5+230 pp. M. 2.60

BEHACKER. See MOČNIK.

BOS (H.). Géométrie élémentaire. 21e édition. Paris, Hachette, 1908. 16mo. 288 pp. Fr. 2.00

DONADT (A.). See LÜBSEN (H. B.).

DUPUIS (J.). Tables de logarithmes à cinq décimales contenant les logarithmes des nombres entiers de 1 à 10000, les logarithmes des sinus et des tangentes calculés de minute en minute jusqu'à 90 degrés, plusieurs tables usuelles et un grand nombre de formules et de nombres utiles. 29e édition. Paris, Hachette, 1908. 16mo. 4+231 pp. Fr. 2.00

FUSS (K.). Resultate, Lösungs-Andeutungen und ausführliche Auflösungen zu den Aufgaben aus der Buchstabenrechnung und Algebra. Für Schulen und zum Selbstunterricht. 6te, vermehrte und verbesserte Auflage. Nürnberg, Korn, 1908. 8vo. 8+325 pp. M. 3.20

GRÉVY (A.). Géométrie théorique et pratique à l'usage de l'enseignement secondaire des jeunes filles (classes de 4e et 5e années). Paris, Vuibert, 1908. 8vo. 197 pp.

HARTENSTEIN (H.). See BARDEY (E.).

HOČEVAR (F.). Lehr- und Uebungsbuch der Geometrie für Untergymnasien. 8te Auflage. Wien, Tempsky, 1907. 8vo. 123 pp. M. 1.80

JACQUEMART (A.). Algèbre pratique en quinze leçons, à l'usage des écoles primaires supérieures et des écoles normales. 5e édition. Paris, Larousse, 1908. 16mo. 112 pp. Fr. 1.00

KRÜNES (F.). See MOČNIK.

LÜBSEN (H. B.). Ausführliches Lehrbuch der ebenen und sphärischen Trigonometrie. Zum Selbstunterricht mit Rücksicht auf die Zwecke des praktischen Lebens bearbeitet. Neu bearbeitet von A. Donadt. 19te Auflage. Leipzig, Brandstetter, 1908. 8vo. 7+146 pp. M. 2.90

MATRICULATION mathematics papers. From Jan. 1897 to Jan. 1908. New edition. London, Clive, 1908. 8vo. 138 pp. 1s. 6d.

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